Tutorial Letter 201/1/2018

Linear Mathematical Programming

DSC2605

Semester 1

Department of Decision Sciences

This tutorial letter contains solutions to Assignment 1

Bar code

Define tomorrow.

UNISA university of south africa

Question 1

Discussion on myUnisa. No solution is provided. [7]

Question 2

Consider the following matrices:

$$A = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & 2 \\ -1 & 3 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & p & p^2 \\ 1 & q & q^2 \\ 1 & r & r^2 \end{bmatrix}$$

(a) Inverse of matrix A: A^{-1} 10

The augmented matrix is given by

$$\left[\begin{array}{ccc|ccc} 2 & 1 & -2 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ -1 & 3 & 1 & 0 & 0 & 1 \end{array}\right]$$

We swap the first and seem d rows, and we obtain

$$\left[\begin{array}{ccc|ccc} \textcircled{1} & 0 & 2 & 0 & 1 & 0 \\ 2 & 1 & -2 & 1 & 0 & 0 \\ -1 & 3 & 1 & 0 & 0 & 1 \end{array}\right]$$

We set our first pivot in the first column and first row and apply the one-zero method. We get:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & \textcircled{1} & -6 & 1 & -2 & 0 \\ 0 & 3 & 3 & 0 & 1 & 1 \end{array}\right]$$

We find our next pivot in the 2^{nd} column and 2^{nd} row. We apply the one-zero method and get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & -2 & 0 \\ 0 & 0 & \textcircled{21} & -3 & 4 & 1 \end{array}\right]$$

The next pivot is found in the third column and third row. We apply the one-zero method to third column and get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{7} & \frac{1}{3} & -\frac{2}{21} \\ 0 & 1 & 0 & \frac{1}{7} & 0 & \frac{2}{7} \\ 0 & 0 & 1 & -\frac{1}{7} & \frac{1}{3} & \frac{1}{21} \end{array}\right]$$

The inverse of the matrix A is then given by

$$A^{-1} = \frac{1}{21}\begin{bmatrix} 6 & 7 & -2 \\ 3 & 0 & 2 \\ -3 & 7 & 1 \end{bmatrix}$$

(b) Determine of the matrix B: |B|

$$\begin{aligned}
|B| &= 1\begin{vmatrix} q & q^2 \\ r & r^2 \end{vmatrix} - p\begin{vmatrix} 1 & q^2 \\ 1 & r^2 \end{vmatrix} + p^2\begin{vmatrix} 1 & q \\ 1 & r \end{vmatrix} \\
&= \left(qr^2 - rq^2\right) - p\left(r^2 - q^2\right) + p^2\left(r - q\right) \\
&= qr^2 - rq^2 - pr^2 + pq^2 + p^2r - p^2q \\
&= (q-p)\left[r^2 - r(q+p) + pq\right] \\
&= (q-p)\left[r^2 - rq - rp + pq\right] \\
&= (q-p)\left[r(r-q) - p(r-q)\right] \\
&= (q-p)(r-q)(r-p)
\end{aligned}$$

(c) Solving $2A - BX = 2I_3$

$$\begin{aligned}
2A - BX &= 2I_3 \\
-2A + 2A - BX &= -2A + 2I_3 \\
BX &= 2\left(A - I_3\right) \\
X &= 2.B^{-1}\left(A - I_3\right)
\end{aligned}$$

If $p = q = r$, then $|B| = 0$ and the inverse of the matrix B does not exist. Consequently there is no solution to the equation $2A - BX = 2I$. If $p \neq q \neq r$, then $|B| \neq 0$ and the inverse B^{-1} exists. We obtain:

$$\text{Cof}(B) = \begin{bmatrix} rq(r-q) & -(r^2-q^2) & r-q \\ -pr(r-p) & r^2-p^2 & -(r-p) \\ pq(q-p) & -(q^2-p^2) & q-p \end{bmatrix}$$

$$\text{Cof}(B)^T = \begin{bmatrix} rq(r-q) & -pr(r-p) & pq(q-p) \\ -(r^2-q) & r^2-p^2 & -(q^2-p^2) \\ r-q & -(r-p) & q-p \end{bmatrix}$$

This implies that

$$B^{-1} = \frac{1}{(r-q)(r-p)(q-p)} \begin{bmatrix} rq(r-q) & -pr(r-p) & pq(q-p) \\ -(r^2-q^2) & r^2-r^2 & -(q^2-p^2) \\ r-1 & -(r-p) & q-p \end{bmatrix}.$$

We also have:

$$\begin{aligned} A - I_3 &= \begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & 2 \\ -1 & 3 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 2 \\ -1 & 3 & 0 \end{bmatrix} \end{aligned}$$

It follows that

$$\begin{aligned} X &= 2B^{-1}(A - I_3) \\ &= \tfrac{2}{(r-q)(r-p)(q-p)} \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned} x_{11} &= (q-p)(r^2 - rp - rq - pq) \\ x_{12} &= rq(r-q) + pr(r-p) + 3pq(q-p) \\ x_{13} &= 2r(p^2q^2 - pr - qr) \\ x_{21} &= 2(q^2 - p^2) \\ x_{22} &= -2(r^2 + q^2 - 2p^2) \\ x_{23} &= -2(p^2 + q^2 - 2r^2) \\ x_{31} &= -2(q-p) \\ x_{32} &= 2(r + q - 2p) \\ x_{33} &= 2(p + q - 2r) \end{aligned}$$

(d) Equivalent matrix B'

$$B' = \begin{bmatrix} 1 - \frac{p}{q} & 0 & p^2 - pq \\ \frac{1}{q} & 1 & q \\ 1 - \frac{r}{q} & 0 & r^2 - rq \end{bmatrix}$$

Question 3

The augmented matrix of the linear system is

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 1 & 0 & 1 & 1 & 4 \\ 1 & 0 & 1 & 1 & 2 \end{array}\right]$$

The first pivot is in the first column and first row. We apply the one-zero method and get the following

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & -1 & 0 & 0 & -2 \\ 0 & -1 & 0 & -1 & -4 \end{array}\right]$$

The next pivot is in the second column and second row. We apply the one-zero method and obtain:

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 4 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & -1 & -2 \end{array}\right]$$

The next pivot is found in the fourth column and third row. After applying the one-zero method to the pivot column, we

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

(a) z is the only one free variable and x, y and w are leading variables.

(b) The system has infinity many solution and the general solution is given by

$$\begin{aligned} x &= 2 - t \\ y &= 2 \\ z &= t \\ z &= t \\ w &= 2 \end{aligned}$$

and the solution set is

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \left\{ \begin{bmatrix} 2-t \\ 2 \\ t \\ 2 \end{bmatrix}, \quad t \in \mathbb{R} \right\}.$$

Question 4

The system is homogeneous if and only if $a = b = c = 0$.

(a) The augmented matrix of the homogenous system is

$$\left[\begin{array}{rrr|r} 1 & 1 & -1 & 0 \\ 1 & -1 & 2 & 0 \\ 3 & 1 & 0 & 0 \end{array}\right]$$

The first pivot is found in the first column and first column we apply the one-zero method to the pivot column. We get

$$\left[\begin{array}{rrr|r} 1 & 1 & -1 & 0 \\ 0 & -2 & 3 & 0 \\ 0 & -2 & 3 & 0 \end{array}\right]$$

The next pivot is in the second column and second row. We apply once more the one-zero method to the pivot column and get

$$\left[\begin{array}{rrr|r} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The general solution is

$$\begin{aligned} x &= -\tfrac{1}{2}t \\ y &= \tfrac{3}{2}t \\ z &= t, \quad t \in \mathbb{R} \end{aligned}$$

To find a nontrivial solution, we need to chose a value of $t \neq 0$. For example if $t = 2$, we get a nontrivial solution given by $x = -1$, $y = 3$, $z = 2$.

(b) The augmented matrix of the linear system

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & a \\ 1 & -1 & 2 & b \\ 3 & 1 & 0 & c \end{array}\right]$$

We find the first pivot on the first column and first row. We apply the one-zero method and get

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & a \\ 0 & -2 & 3 & b-a \\ 0 & -2 & 3 & c-3a \end{array}\right]$$

The next pivot is found in the second column and second row. We apply the one-zero method once more and get:

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & \frac{a-b}{2} \\ 0 & 1 & -\frac{3}{2} & \frac{b-a}{2} \\ 0 & 0 & 0 & -2a-b+c \end{array}\right]$$

(b_1) The system has no solution for all $a, b, c \in \mathbb{R}$ such that $-2a - b + c \neq 0$

(b_2) The system will never have a unique solution.

(b_3) The system has infinitely many solution for all $a, b, c \in \mathbb{R}$ such that

$$\begin{aligned} -2a - b + c &= 0 \\ \text{or} \quad 2a + b - c &= 0 \end{aligned}$$