COS3761 Memo 2017

QUESTION 1 [28]

Question 1.1

Consider the following propositional symbols and their intended meanings:

Propositional symbols	Meanings
p	Bob needs a parachute
q	Bob plans to jump from the plane
r	The plane is flying at 25000 feet
s	Bob needs an oxygen mask

(i) Express the following declarative sentence in propositional logic using the propositional symbols as given above:

Bob does not need a parachute unless he plans to jump from the plane and the plane is flying at 25000 feet.

$(q \land r) \rightarrow \neg p$ OR $p \rightarrow \neg (q \land r)$ (2)

(ii) Express the following propositional logic formula in English where the propositional symbols p, q, r and s have the meanings given above:

$(r \land q) \rightarrow (p \land s)$ (2)

If the plane is flying at 25000 feet and Bob plans to jump from the plane then Bob needs a parachute and an oxygen mask.

Question 1.2

Use the basic natural deduction rules for propositional logic to prove the validity of the following sequents

(i) $(p \wedge q) \rightarrow r \vdash p \rightarrow (q \rightarrow r)$ (5)

1	**$(p \wedge q) \rightarrow r$**	**premise**
2	**p**	**assumption**
3	**q**	**assumption**
4	**$p \wedge q$**	**$\wedge$ i 2, 3**
5	**r**	**$\rightarrow$ e 1, 4**
6	**$q \rightarrow r$**	**$\rightarrow$ i 3 – 5**
7	**$p \rightarrow (q \rightarrow r)$**	**$\rightarrow$ i 2 – 6**

(ii) $p \wedge (q \vee r) \vdash (p \wedge q) \vee (p \wedge r)$ (10)

1	**$p \wedge (q \vee r)$**	**premise**
2	**p**	**$\wedge$ e 1**
3	**$q \vee r$**	**$\wedge$ e 1**
4	**q**	**assumption**
5	**$p \wedge q$**	**$\wedge$ i 2, 4**
6	**$(p \wedge q) \vee (p \wedge r)$**	**$\vee$ i 5**
7	**r**	**assumption**
8	**$p \wedge r$**	**$\wedge$ i 2, 7**
9	**$(p \wedge q) \vee (p \wedge r)$**	**$\vee$ i 8**
10	**$(p \wedge q) \vee (p \wedge r)$**	**$\vee$e 3, 4-6, 7-9**

Question 1.3

Show that the following sequent is not valid by giving an appropriate valuation.

$(\neg p \lor \neg q) \lor r, q \lor r, p \vdash \neg r$

(4)

Explain why your valuation proves that the sequent is not valid.

p	**q**	**r**	**¬ p ∨ ¬ q ∨ r**	**q ∨ r**	**¬ r**
F	**F**	**F**	**T**	**F**	**T**
F	**F**	**T**	**T**	**T**	**F**
F	**T**	**F**	**T**	**T**	**T**
F	**T**	**T**	**T**	**T**	**T**
T	**F**	**F**	**T**	**F**	**F**
T	**F**	**T**	**T**	**T**	**F**
T	**T**	**F**	**F**	**T**	**T**
T	**T**	**T**	**T**	**T**	**F**

To show that a sequent is not valid, we must find a valuation in which the formulas on the left-hand side are true, but in which the formula on the right-hand side is false.

Question 1.4

Use the HORN algorithm to prove that the following Horn formula is satisfiable. Show each step.

$(T\to q) \land (T \to s) \land (w \to\bot) \land (p \land q\land s\to v) \land (v\to s) \land (T\to r) \land (r \to p)$ (5)

Solution:

- $(\underline{T}\to q) \land (\underline{T} \to s) \land (w \to\bot) \land (p \land q \land s\to v) \land (v\to s) \land (\underline{T}\to r) \land (r \to p)$
- **Mark: q, s, r through (T→q), (T→s),(T→r)**
- $(\underline{T}\to \underline{q}) \land (\underline{T} \to \underline{s}) \land (w \to\bot) \land (p \land \underline{q}\land \underline{s}\to v) \land (v\to \underline{s}) \land (\underline{T}\to \underline{r}) \land (\underline{r} \to p)$
- **Mark: p through (r→p)**
- $(\underline{T}\to \underline{q}) \land (\underline{T} \to \underline{s}) \land (w \to\bot) \land (\underline{p} \land \underline{q}\land \underline{s}\to v) \land (v\to \underline{s}) \land (\underline{T}\to \underline{r}) \land (\underline{r} \to \underline{p})$
- **Mark: v through** $(\underline{p} \land \underline{q} \land \underline{s}\to v)$
- $(\underline{T}\to \underline{q}) \land (\underline{T} \to \underline{s}) \land (w \to\bot) \land (\underline{p} \land \underline{q}\land \underline{s}\to \underline{v}) \land (\underline{v}\to \underline{s}) \land (\underline{T}\to \underline{r}) \land (\underline{r} \to \underline{p})$

Nothing more can be marked, and ⊥ hasn't been marked, so the original Horn formula is satisfiable.

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QUESTION 2 [38]

Question 2.1

Consider the following predicate, constant symbols and their intended meanings:

T(x)	x has been cashed within 10 days
C(x)	x is a cheque
V(x)	x is void
S(x, y)	x can cash y
l	Liza
b	Briz

(i) Express the following predicate logic formula in English, where the symbols have the meanings as given above:

$\forall x\ ((C(x) \ \wedge \ \neg T(x)) \rightarrow V(x))$ (3)

A cheque is void if it has not been cashed for 10 days

(ii) Express the following declarative sentence in predicate logic using the symbols as given above:

Liza and Briz cannot cash a cheque which is void. (3)

$\forall x\ ((C(x) \ \wedge V(x)) \rightarrow (\neg S (l, x) \wedge \neg S(b,x)))$

Question 2.2

Say

- F, G are predicate symbols with one argument and P is a predicate with two arguments.
- x, y are variables.
- f is a function with one argument
- b, c are constants.

State which of the following are well formed formulas: (5)

(i) $f(c) \rightarrow \forall x \forall y P(x, y)$ Not a wff. The term f(c) can only be an argument of a predicate in a wff.

(ii) P (f(x), b) wff

(iii) **(G(x) → F(x))** **wff**

(iv) **G (P(x, y))** **Not a wff. Predicate may not be an argument of a predicate.**

(v) **∀x F(y) ∧ ∃y G(x)** **wff**

Question 2.3

Consider the following formula ϕ where P and Q are predicate symbols with one argument.

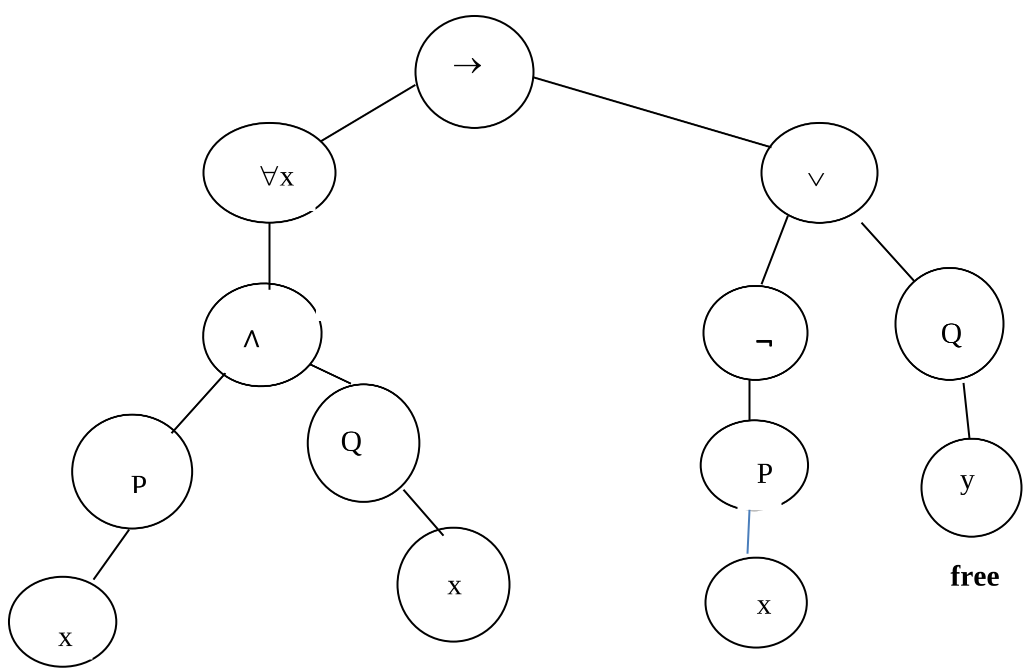
$$(\forall x(P(x) \land Q(x))) \rightarrow (\neg P(x) \lor Q(y))$$

(i) Draw the parse tree of ϕ. (5)

(ii) Mark the free and bound variables on the tree. (2)

(iii) Let f be a function with one argument. Is f(x) free for y in ϕ? Explain your answer. (3)

(i)



(ii) **bound** **bound** **free**

(iii) **Yes. The free occurrence of y does not fall within the scope of ∀x.**

Question 2.4

Using the basic natural deduction rules for predicate logic, prove the validity of the following sequent:

(9)

$\forall x((P(x) \wedge Q(x)) \rightarrow R(x)), \exists x(Q(x) \wedge \neg R(x)) \vdash \exists x(\neg P(x))$

1		$\forall x(P(x) \wedge Q(x) \rightarrow R(x))$	**premise**
2		$\exists x(Q(x) \wedge \neg R(x))$	**premise**
3	**a**	$Q(a) \wedge \neg R(a)$	**assumption**
4		$Q(a)$	$\wedge$ **e 3**
5		$\neg R(a)$	$\wedge$ **e, 3**
6		$P(a) \wedge Q(a) \rightarrow R(a)$	$\forall$ **e 1**
7		$P(a)$	**assumption**
8		$P(a) \wedge Q(a)$	$\wedge$ **i 4, 7**
9		$R(a)$	$\rightarrow$ **e 6, 8**
10		$\perp$	$\neg$ **e 5, 9**
11		$\neg P(a)$	$\neg$ **i 7-10**
12		$\exists x(\neg P(x))$	$\exists x$ **i 11**
13		$\exists x(\neg P(x))$	$\exists x$ **e 2, 3-12**

Question 2.5

Let ϕ be the following formula:

$$\forall x \forall y (R(x, y) \rightarrow R(x, f(y)))$$

(i) Show that ϕ is satisfiable by constructing a mathematical model (where the universe *A* of concrete values is the set of integers) Explain why your model satisfies ϕ. (4)

(ii) Show that ϕ is not valid by constructing a mathematical model (where the universe *A* of concrete values is the set of integers). Explain why your model falsifies ϕ. (4)

(i) Model M where the sentence is true:

A : the set of integers (given)

R^M **: We interpret R(x, y) as "x > y" (or …)**

f^M**: We interpret f(x) as "x-10" (or …)**

In this interpretation the formula states that for all integers x and y, if x > y then x > y-10. which is obviously true.

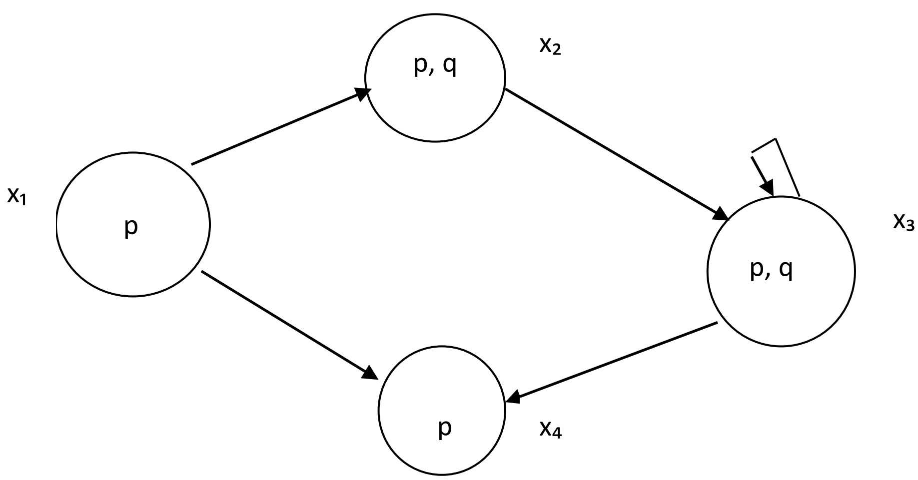
(ii) Model M where the sentence is false:
A : the set of integers (given)
R^M : We interpret R(x, y) as "x = 2 times y"(or …)
f^M: We interpret f(x) as "x-10" (or …)

In this interpretation the formula states that for all integers x and y, the formula is not valid in this interpretation since there are values for ax and y for which the formula is false. For example x = 8 and y = 4. the antecedent namely R(x, y) is true since 8 = 2*4 . However the consequent namely R(x, f(y)) is false , since 8 is not equal to 4 – 10= (- 6)

QUESTION 3 [34]

Question 3.1

Consider the following Kripke model with worlds x_1, x_2, x_3 and x_4:



For each of the following relations, determine whether it holds in the above Kripke model and give reasons for your answer: (12)

(i) $x_1 \Vdash \Box p$

(ii) $x_2 \Vdash \Box (p \wedge q)$

(iii) $x_3 \Vdash \Box p \wedge \Diamond q$

(iv) $x_4 \Vdash \Box \Box \neg p$

(i). **The formula is true in world x_1 because p is true in all the worlds accessible from x_1 namely x_2 and x_4. Therefore $\Box$ p is true in x_1.**

(ii). The formula is true in world x_2 because $p \wedge q$ is true in x_3, and since x_3 is the only world accessible from x_2. Therefore $\Box (p \wedge q)$ is true in x_2.

(iii). The formula is true in world x_3 because $\Box p$ is true in x_3 since p is true in all worlds accessible from x_3, namely x_3 and x_4. Also $\Diamond q$ is true in x_3 since x_3 is accessible to itself.

(iv). The formula is true in world x_4 because there are no worlds accessible from x_4.

Question 3.2

Suppose we want to re-engineer basic modal logic to fit the following reading of $\Box \phi$:

$\Box \phi$: The agent believes that ϕ.

For each of the following two formulas, state whether it should be valid or not and explain your answer in both cases.

(i) $(\Box \phi \rightarrow \phi) \vee (\phi \rightarrow \Box \phi)$ (3)

This should not be valid: If the agent believes ϕ, it need not really be true, and if ϕ is true, it is possible that the agent does not believe it

(ii) $\Box \neg \phi \rightarrow \Box \Box \neg \phi$ (3)

This should be valid: If the agent believes $\neg \phi$, the agent believes that he believes $\neg \phi$.

Question 3.3

Say we interpret the modal operators $\Box$ and $\Diamond$ to represent the temporal notions "Always" and "sometimes" respectively, and say we interpret the propositional letters p and q to mean "It rains today" and "There will be sudden floods" respectively.

(i) Express the following modal logic formula in English:

$\neg \Box (p \rightarrow q)$ (2)

It is not always true that if it rains today, there will be sudden floods.

(ii) Translate the following declarative sentence into modal logic:

Even though it doesn't rain today sometime in the future there will be sudden floods.

$\neg \; p \land \Diamond q$ **(2)**

Question 3.4

(i) Express the following sentence in modal logic where K_i is read as "Agent i knows that":

Agent 1 knows p but he doesn't know that agent 2 knows q (3)

$K_1 \, p \land \neg K_1 K_2 \, q$

(ii) Express the following modal logic formula in English where K_i is read as "Agent i knows that":

$K_1 K_2 \neg q \rightarrow p$ (3)

If agent 1 knows that agent 2 knows not q, then p

Question 3.5

Using the basic natural deduction rules, the □ introduction and elimination rules of the basic modal logic K as well as the following three additional rules for KT45

$$\frac{\Box \phi}{\phi}\,T \qquad \frac{\Box \phi}{\Box \Box \phi}\,4 \qquad \frac{\neg \Box \phi}{\Box \neg \Box \phi}\,5$$ (6)

prove the validity of the following sequent:

$\Box (p \rightarrow q), \;\; \Box \neg q \;\vdash\; \Diamond \neg p$

We write $\Diamond$ as $\neg \Box \neg$. So we have to prove the following:

$\Box (p \rightarrow q), \;\; \neg \Box q \;\vdash\; \neg \Box \neg \neg p$

or, equivalently:

$\Box (p \rightarrow q), \;\; \neg \Box q \;\vdash\; \neg \Box p$

1	**$\Box (p \rightarrow q)$**	**premise**
2	**$\neg \Box q$**	**premise**
3	**$\Box p$**	**assumption**
4	**p**	**$\Box$e 3**
5	**$p \rightarrow q$**	**$\Box$e 1**
6	**q**	**$\rightarrow$e 4, 5**
7	**$\Box q$**	**$\Box$i 4 - 6**
8	**$\bot$**	**$\neg$e 7, 2**
9	**$\neg \Box p$**	**$\neg$i 3 - 8**