Tutorial letter 103/3/2017

Formal Logic 2

COS2661

Semester 1 & 2

School of Computing

IMPORTANT INFORMATION:

Mathematical Background

BAR CODE

Learn without limits.

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CONTENTS

CHAPTER 1 NUMBERS

Section 1: Natural numbers

1.1 The set

The set of natural numbers is the set

N = {0, 1, 2, 3, ...}.

To say that the natural number n is an element of the set of natural numbers, we write $n \in$ N. By writing 3, 5 $\in$ N, we mean "three and five are both elements of N (the set of natural numbers)".

The set may be visualised as follows:

.	.	.	.	.	.	.
0	1	2	3	4		

Important features of this set:

- it is an infinite set, in other words it has no last element
- it has a starting point (smallest element), namely zero
- each element in this set has a successor, and each element, except for zero, has a predecessor
- the natural numbers are used to count elements:

⌂	⌂	⌂	⌂
1	2	3	4

1.2 The operations

Introduction

The operations of adding

$a + b$; $a, b \in$ N

and multiplication

$a \times b$ or $a \cdot b$ or $a(b)$ or ab or $(a)b$ or $(a)(b)$; $a, b \in$ N

are unlimitedly executable in the set of natural numbers. This means that the answer of a sum (the result of adding numbers) or product (the result of multiplying numbers) of two or more natural numbers will also always be a natural number, in other words, the set of natural numbers is closed under addition and multiplication.

The operations of subtracting

$a - b$; $a, b \in \mathrm{N}$

and division

$a \div b$ or a/b; $a, b \in \mathrm{N}$

are, however, not unlimitedly executable, which implies that the answer of the difference (result of subtracting numbers) or quotient (result of dividing numbers) of two natural numbers will not always be another natural number.

Examples:

$5 – 7 \notin \mathrm{N}$

$2 \div 4 \notin \mathrm{N}$

Laws for the operations on N

- Addition and multiplication are *commutative* in the set of natural numbers, that is, the sequence in which the numbers are combined makes no difference to the answer of the sum or product: $a \times b = b \times a$ and $a + b = b + a$; $a, b \in \mathrm{N}$.
- The *associative* law is valid for multiplication and addition, in other words, it doesn't make any difference to the answer of the sum or product of three natural numbers, $a + (b + c) = (a + b) + c$ and $a(bc) = (ab)c$; $a, b, c \in \mathrm{N}$, whether the middle term is associated with the first or with the last term.
- Finally, the *distributive* laws are also valid in the set of natural numbers, that is $a(b + c) = ab + ac$; $a, b, c \in \mathrm{N}$, and also $a(b - c) = ab - ac$; $a, b, c \in \mathrm{N}$, ***but*** the latter only if $b - c \in \mathrm{N}$.

Examples:

(1) $3 + 2 = 5 = 2 + 3$ (Commutative law for addition of natural numbers)

(2) $7 \times 8 = 56 = 8 \times 7$ (Commutative law for multiplication of natural numbers)

(3) $4 - 2 \neq 2 - 4$

(4) $5 \div 3 \neq 3 \div 5$

(5) $(5 + 3) + 2 = 10 = 5 + (3 + 2)$ (Associative law for addition of natural numbers)

(6) $(2 \cdot 3) \cdot 4 = 24 = 2 \cdot (3 \cdot 4)$ (Associative law for multiplication of natural numbers)

(7) $(6 - 3) - 3 = 0 \neq 6 - (3 - 3)$

(8) $(5 \div 2) \div 2 \neq 5 \div (2 \div 2)$

(9) $2 \cdot (3 + 6) = 2 \cdot 3 + 2 \cdot 6 = 18$ (Distributive law for addition and multiplication of natural numbers)

(10) $4 \div (6 - 1) \neq (4 \div 6) - (4 \div 1)$

1.3 Ordering

Introduction

- We use the symbols "<" and "≤ " to study the *ordering* of the set of natural numbers. The two symbols mean "is smaller than" and "is smaller than or equal to", respectively.
- $n \leq m$ and $m \geq n$ mean the same thing.

Examples:

$3 < 4$

$3 \leq 4$

$3 \leq 3$

Relations between operations and ordering in N

$n \leq m \;\Leftrightarrow\; (\exists k \in \mathrm{N})\,(n + k = m)$

If $n \leq m$, then $(\forall k \in \mathrm{N})\,(n + k \leq m + k)$ and $(\forall k \in \mathrm{N})\,(n \times k \leq m \times k)$

Questions:

Suppose $n \leq m$. Is $n - k \leq m - k$, $\forall k \in \mathrm{N}$?

Is $n \div k \leq m \div k$, $\forall k \in \mathrm{N}$?

1.4 Factors and multiples

The *factors* of a natural number are those natural numbers by which that number can be divided (without a remainder).

A *multiple* of a natural number is the number created by multiplying that number with any natural number.

Examples:

The set consisting of the factors of 18 is {1, 2, 3, 6, 9, 18}

The set consisting of the multiples of 18 is {0, 18, 36, 54, 72, ...}

1.5 Prime numbers

A prime number is a natural number bigger than 1 which has only itself and 1 as factors. Examples of prime numbers are 2, 3, 5, 7, 11, ... Prime factors are those factors of any number that are prime numbers. By prime factorisation is meant the fact that each natural number bigger than 1 has a unique factorisation as the product of powers of prime numbers: $n = p_1^i p_2^j ... p_k^m$, where the prime bases p_1, p_2, ..., p_k and the exponents i, j, ..., m are unique.

Examples:

The prime factors of 18 are 2 and 3.

The prime factorisation of 18 is 2×3^2.

Section: 2 Integers

2.1 The set

Introduction

The set of integers is the set

I = {..., -3, -2, -1, 0, 1, 2, 3, ...}.

In other words, it is the set N = {0, 1, 2, 3, ...} together with the set of negative numbers {-1, -2, -3, ...}. Thus the set can be divided into non-negative integers (i.e. the positive integers and zero) and the negative integers.

The set may be visualised as follows:

. . . .

-3 -2 -1 0 1 2 3 4

Other features:

- the set is infinite in two directions (positive and negative)
- it can, however, be counted in the following way:

7 5 3 1 2 4 6 8

. . . .

-3 -2 -1 0 1 2 3 4

The function / role of negative numbers

We quite often make use of integers in everyday life, for example when reading temperature, when calculating debt, when talking about height above sea level, and so on.

Example:

Say the following temperatures (in degrees Celsius) were measured in certain cities on a winter's morning in July:

Johannesburg: 2

Ladysmith: -1

Durban: 9

Then we can say that Ladysmith was the coldest of the three cities on that morning, while Durban was the hottest.

2.2 Features of zero and one

(1) An integer remains unchanged if zero is added to it. Zero is called the *identity element of addition*.

(2) An integer remains unchanged if zero is subtracted from it.

(3) If any integer is multiplied by zero, the product is zero.

(4) If zero is divided by any integer, the quotient is zero.

(5) Division by zero is undefined. Why? If $8/0$ = something, then $8 = 0 \times$ something, which is impossible since $8 \neq 0$.

(6) Any integer multiplied by one, remains unchanged. One is the *identity element of multiplication*.

(7) Any integer divided by one, remains unchanged.

2.3 The operations

The operations multiplication, addition and subtraction are unlimitedly executable in the set of integers. Only division is not unlimitedly executable in the set of integers. Two integers that are at the same distance from zero on the number line are called each other's additive inverses.

The sum of an integer and its additive inverse is zero. The easiest way in which to find some integer's additive inverse is to change its sign or directedness (positive or negative).

Example:

> The additive inverse of 5 is -5 and the additive inverse of -7 is 7. Note that 5 < 7, while -7 < -5.

Addition in the set of integers

- Addition of two positive numbers

Sue jogged 3km, rested for a few minutes and then jogged another 6km. In total she jogged 9km.

. . . .

0 1 2 3 4 5 6 7 8 9 10 11

3 + 6 = 9

- Addition of two negative numbers

Sue borrows R1. She then borrows another R4. She has thus borrowed a total of R5 today.

. . . .

-6 -5 -4 -3 -2 -1 0 1 2 3 4

(-1) + (-4) = -5

- Addition of a positive to a negative number

This morning the temperature measured -4 degrees Celsius. Since then the temperature has risen by 7 degrees. Thus the temperature is now 3 degrees Celsius.

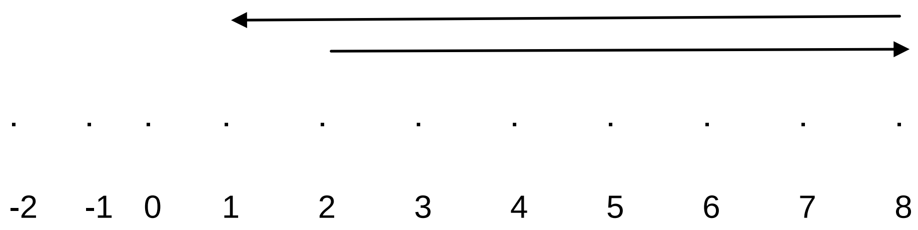
. . . .

-4 -3 -2 -1 0 1 2 3 4 5

$$(-4) + (7) = 3$$

- Addition of a negative to a positive number

It was 6 degrees Celsius early last night. During the night the temperature dropped by 7 degrees. Thus the temperature is now -1 degrees Celsius.



$$6 + (-7) = -1$$

Subtraction in the set of integers

It is necessary to make a few comments on the ambiguity of the minus sign (“-”). There are three interpretations of this sign possible:

- it can denote the negative sign or directedness of an integer, in the way that -7 refers to the negative number -7 (sign or directedness refers to whether a number is positive or negative)
- it can denote a unary operation, specifically the additive inverse $-a$, $\forall a \in$ I (in this context, it has to be noted that $-k$, $k \in$ I, does not necessarily represent a negative number, because the sign or directedness of $-k$ depends on the sign or directedness of k itself: if k is a positive integer, the sign or directedness of $-k$ will be negative, but if k is a negative integer, the sign or directedness of $-k$ will be positive)
- it can denote the binary operation of subtraction $a - b = a + (-b)$, $a, b \in$ I (subtracting some integer b means adding its addition inverse $-b$).
  (Note that the operation symbol plus (“+”) is only needed as denoting a binary operation. It can be used for positive sign or directedness, but is superfluous because +7 = 7.)

Multiplication and division in the set of integers

In general for any integers m and n:

- The product of two integers with the same sign or directedness is positive.
- The product of two integers of opposite sign or directedness is negative.
- When two integers of the same sign or directedness is divided, the sign or directedness of the quotient is positive.

- When two integers with opposite sign or directedness are divided, the sign or directedness of the quotient is negative.

Examples:

(1) $-3 \times -4 = 12$ and $6 \times 7 = 42$

(2) $-3 \times 4 = -12$ and $6 \times -7 = -42$

(3) $-8 \div -2 = 4$ and $15 \div 5 = 3$

(4) $-8 \div 2 = -4$ and $15 \div -5 = -3$

Laws of operations in the set of integers

The commutative, associative and distributive laws are valid for the set of integers too, as they are for the set of natural numbers.

2.4 Ordering

Note that if $k < m$; $k, m \in$ I, then k lies “left of” m on the number line. Note also that $-3 < -1$, even though $1 < 3$.

. . . .

-4 -3 -2 -1 0 1 2 3 4

Relations between operations and ordering

$(-k \leq -m) \Leftrightarrow (m \leq k)$, $\forall k, m \in$ I, is a specific example which illustrates the more general case where multiplication by a negative number leads to “inversion” of the “is smaller than or is equal to” sign (“$\leq$”), and similarly for <.

Examples:

$-3 < -1$ but $3 > 1$

$-3 \leq 5$ but $(-2)(-3) \geq (-2)\,5$

Section 3: Rational numbers

3.1 The set

Introduction

Rational numbers or fractions are all numbers of the form given in the definition of Q, the set of rational numbers:

$$Q = \{\tfrac{p}{q} \mid p, q \in I,\ q \neq 0\}$$

The top integer, p, of a fraction $\frac{p}{q}$ is called its numerator, and the bottom integer, q, is called its denominator.

In the following we will sometimes write the rational number $\frac{p}{q}$ as p/q.

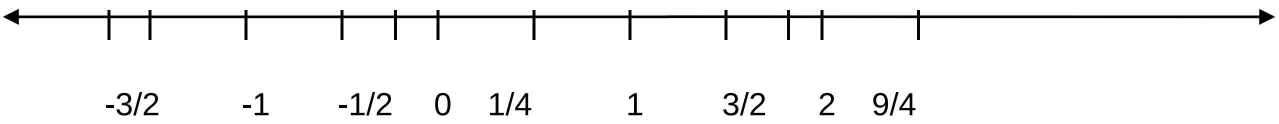
Examples:

$\frac{4}{5}$

$\frac{4}{53}$

$\frac{-1}{2} = \frac{1}{-2} = -\frac{1}{2}$

We may visualise the number line and a few rational numbers as follows:



Note that every rational number is either equal to an integer, or lies between two integers on the number line. The set of integers is a subset of the set of rational numbers, because $k = \frac{k}{1}$, $k \in I$, and thus the set of rational numbers is an extension of the set of integers, as the set of integers is an extension of the set of natural numbers.

Equivalent fractions

Example:

16/24 = 8/12 = 4/6 = 2/3 = -2/-3 = -16/-24

We divide or multiply the numerator and denominator of some fraction with the same non-zero number in order to calculate the fractions equivalent to this fraction.

Use is made of calculating equivalent fractions when fractions are compared, as well as in all the operations in the set of rational numbers.

Decimal numbers

The rational numbers or fractions that have been discussed thus far are sometimes called "ordinary" fractions.

Because addition and subtraction of ordinary fractions can sometimes become very tedious, ordinary fractions are sometimes written as decimal fractions. The fraction 3/4 can for instance be written as 0,75. The comma denotes that the numbers following it are tenths, hundredths and so on (in powers of one tenth). Some (especially American) writers use the decimal point instead of the comma: 0.75.

When an ordinary fraction has to be rewritten as a decimal fraction, it is sometimes possible to change the fraction to an equivalent fraction with the denominator written as a power of ten. Then it is easy to write down the equivalent decimal fraction: simply move the decimal comma as many positions to the left as the relevant power of ten.

Example:

$$3/4 \;=\; 75/100 \;=\; 75{,}0/100 \;=\; 0{,}75$$

In other cases (when we are working with decimal numbers that have an infinitely long decimal fraction, such as 1/3 = 0,33333333...) this is not possible, but then the fraction is written as the sum of fractions of which the denominators are powers of ten.

Example:

$$1/3 \;=\; 3/10 + 3/100 + 3/1000 + 3/10000 + \ldots$$

Formally: The decimal number

$k{,}d_1d_2\ldots,\;\; k \in \mathrm{I};\;\; d_i \in (0,1,2,3,4,5,6,7,8,9\}$

may be written as follows:

$k{,}d_1d_2\ldots \;=\; k + d_1/10 + d_2/100 + d_3/1000 + \ldots$

Here "k" is called the *integer part* and $0{,}d_1d_2\ldots$ the *decimal fraction* of this decimal representation or expansion. The comma thus separates the integers from the fractions.

Examples:

473 = 400 + 70 + 3 = 4(100) + 7(10) + 3(1)

0,543 = 5(1/10) + 4(1/100) + 3(1/1000)

Recurring fractions

A decimal expansion can be found for any decimal number by "long division" as shown in the following example.

Example:

The decimal expansion of 3/8:

```
    0,375
   ______
 8| 3
    0
   --
    30
    24
   ---
      60
      56
     ---
        40
        40
       ---
         0
```

So, 3/8 = 0,375

The process of division ends when the remainder is zero. However, if the process is not terminating - that is, if the remainder never becomes zero - a repetitive pattern will be created. (This is exactly where the difference between rational and irrational numbers lies - more about this later.)

Example:

The decimal expansion of 1/9:

$$\begin{array}{r|l} & 0{,}11\ldots \\ \cline{2-2} 9 & 1 \\ & \underline{0} \\ & 10 \\ & \underline{\ \ 9} \\ & \ \ 10 \\ & \ \ \underline{\ \ 9} \\ & \ \ \ \ 10 \end{array}$$

1/9 is written as $0{,}\dot{1}$ and we read "zero comma one recurring".

- $k{,}d_1\dot{d}_2 = k{,}d_1d_2\ d_2\ d_2\ d_2\ldots$ and we (for example) read $0{,}1\dot{3}$ as "zero comma one followed by three recurring".
- $k{,}\dot{d}_1\dot{d}_2 = k{,}d_1d_2\ d_1\ d_2\ d_1\ d_2\ d_1\ d_2\ldots = k{,}\overline{d_1d_2}$ and we (for example) read $0{,}\overline{14}$ as "zero comma one four both recurring".

3.2 The operations

Addition and multiplication in the set of rational numbers

The following is valid in general:

$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$; and $\frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}$; $a, b, c \in$ I and $b \neq 0$

Steps when adding and subtracting fractions:

- Write all "mixed numbers" (e.g. 5 + 1/2) as "improper fractions. Thus 5+1/2 is written as $\frac{11}{2}$:

$$5 + \frac{1}{2} = \frac{10}{2} + \frac{1}{2} = \frac{10+1}{2} = \frac{11}{2}$$

(Note: Even mathematicians are not always consequent in their notation! If we write $5\frac{1}{2}$, we usually mean $5+\frac{1}{2}$ and not $5 \times \frac{1}{2}$.)

- Make the denominators of all the fractions the same by calculating the least common multiple of all the denominators in the example and then calculating each fraction's equivalent with that special LCM as denominator. Note that rational numbers can only be added or subtracted if they are written in such a way that all their denominators are the same.
- Calculate the sum or difference of the numerators of these fractions. Note that $\frac{7+5+3}{2}$ = 7/2 + 5/2 + 3/2.
- If you like, convert improper fractions to mixed numbers.

Examples:

2/5 + 3/2 = 4/10 + 15/10 = 19/10 [= (10+9)/10 = 10/10 + 9/10 = 1 + 9/10]

$1\frac{1}{2}$ - 1/3 = 3/2 - 1/3 = 9/6 – 2/6 = (9-2)/6 = 7/6 [= (6+1)/6 = 6/6 + 1/6

= 1 + 1/6]

Multiplication and division in the set of rational numbers

The following is valid in general:

$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$; $b, d \neq 0$ and $a, b, c, d \in \mathrm{I}$

and

$\frac{p}{q} \div \frac{r}{s} = \frac{p}{q} \times \frac{s}{r} = \frac{ps}{qr}$; $q, r, s \neq 0$ and $p, q, r, s \in \mathrm{I}$

Multiplication in the set of rational numbers has a special feature: for any fraction except zero, there exists another fraction such that their product is one. These two numbers are called each other's multiplicative inverses:

$p/q \times q/p = 1$; $p, q \neq 0$.

Steps when multiplying and dividing fractions:

- Rewrite as improper fractions.
- Simplify fractions where possible (to more simple equivalent fractions). Note: When simplifying, the numerator and the denominator are divided by the *same* number.
- Multiply numerators with numerators and denominators with denominators when multiplying.
- Dividing by a fraction means multiplying by its multiplicative inverse.

Examples:

Multiply 11/17 with 17/11: 11/17 × 17/11 = 1

Divide 1/2 by 3/2: 1/2 ÷ 3/2 = 1/2 × 2/3 = 1/3

Multiply 8/9 with 4⅓: 8/9 × 4⅓ = 8/9 × 13/3 = 104/27

Divide 11/15 by 1/3: 11/15 ÷ 1/3 = 11/15 × 3/1 = 33/15

Laws for operations in the set of rational numbers

The laws for operations valid in the sets of natural numbers and integers apply in the set of rational numbers too.

The set of rational numbers are of course closed for all four operations, with division by zero not allowed.

3.3 Decimal numbers

The following is valid for finite decimal calculations.

Addition and subtraction of decimal numbers

The only thing to keep in mind here is to keep the place values directly beneath each other.

Examples:

6,4 – 2,75:

```
 6,40
-2,75
 ----
 3,65
```

20 + 2,16 + 4,701:

```
 20
 2,16
+4,701
 ------
 26,861
```

Multiplication and division by powers of 10

- To multiply a decimal fraction by a power of ten, move each number one position to the left (or if you prefer, the decimal comma one position to the right) for the number of places corresponding to the exponent of the power of ten.

- To divide a decimal fraction by a power of ten, change the positions of the numbers as for multiplication, only now to the right (that is, the decimal comma to the left).

Examples:

0,06 × 10: 6/100 × 10/1 = 6/10 = 0,6

2,35 ÷ 10: 235/100 ÷ 10/1 = 235/100 × 1/10 = 235/1000 = 0,235

Multiplication and division

- The only thing to keep in mind when multiplying is to place the comma in the correct position in the answer to the calculation. This position is determined by the sum of the number of decimal places of all the numbers being multiplied.

Example:

$$\begin{array}{r} 0{,}285 \\ \times \quad 3{,}2 \\ \hline 570 \\ 8550 \\ \hline 0{,}9120 \end{array}$$

since there were 3 + 1 = 4 positions after the decimal commas.

- When doing division, the only thing to keep in mind is to keep track of the place values correctly.

Example:

448 ÷ 0,32 = 448/0,32 = 448/0,32 × 100/100 = 44800/32 = 1400

Note: Operations with infinite decimal expansions are more easily executable if the expansions are written as ordinary fractions via the method set out in the examples in Section 4.1 below.

3.4 Ordering

For a, b, c, $d \in$ I; b, $d \neq 0$; b and d positive integers,

$$a/b \leq c/d \Leftrightarrow ad \leq bc$$

where the right-hand statement is a statement about integers.

(Think about the above as multiplying both fractions by the positive number *bd*.)

In the same way, but, for precisely one of b and d a negative number, $a/b \leq c/d \Leftrightarrow ad \geq bc$.

(See the section concerning operations in the set of integers again.)

Note:

$-1000 \leq 1/1000$, since -1000 is left of 1/1000 on the number line.

Section 4: Real numbers

4.1 The set

Every real number is either rational or irrational. The real numbers form the (complete) number line.

-3 -2 -1 0 1 2 3

Rational numbers have a finite decimal expansion or an infinite recurring decimal expansion. In the three examples below we show how these expansions may be written as fractions.

Examples:

(1) Finite expansion: $4{,}52 = 4 + 52/100 = (400 + 52)/100 = 452/100$

(2) Infinite recurring expansion: $x = 3{,}\dot{1}$

$$\begin{array}{ll} 10x - x: & 31{,}111... \\ & -\ 3{,}111... \\ \hline & 28{,}000... \end{array}$$

Thus $9x = 28$

Thus $x = 28/9$

(3) Infinite recurring expansion: $z = 2{,}51\overline{321} = 2{,}51 + 0{,}00\overline{321}$

Let $x = 0{,}00\overline{321}$

$$\begin{array}{ll} \text{Then } 1000x - x: & 3{,}21321321321... \\ & -0{,}00321321321... \\ \hline & 3{,}21000000000... \end{array}$$

Thus $999x = 3,21$

Thus $x = 3,21/999 = 321/99900$

Thus $z = 2,51 + x$

$= 251/100 + 321/99900$

$= (999 \times 251 + 321)/99900$

- **Irrational numbers**, however, have infinite *non-repetitive* decimal expansions. An example of an irrational number is the square root of 2 (i.e. $\sqrt{2}$).

Countable: A set is countable if the set is finite (i.e. if it has a finite number of elements), or if we can associate positive integers with the set in such a way that every element in the set corresponds to a unique positive integer, and vice versa, that with every positive integer there corresponds a unique element of the set.

- The set of integers is countable. See Section 2.1.
- The set of all rational numbers is countable
- The set of real numbers is, however, not countable.

4.2 Operations

The set of all real numbers is closed under all four operations, except for division by zero. It is an extension of the set of rational numbers. It includes numbers such as $\sqrt{2}$ and calculating square roots is now possible for all non-negative real numbers. The calculation of the square roots of negative numbers is however still not allowed, that is, numbers such as $\sqrt{-2}$ can still not be treated in the set of real numbers. (These numbers are in the set of complex numbers.)

Operations have properties as set out in Sections 1 to 4.

CHAPTER 2 SETS

Section 1: Sets, their elements and equality

The intuitive idea behind sets is the grouping of mathematical objects (for example numbers) in a unity (set, class, family, collection). The objects are called elements of the set. An example is I, the set of all integers - every integer is an element of I. Therefore 5, -7, 11 all are elements of I.

The elements of a set completely determine the content of the set - two sets with the same elements are actually one and the same set. For example, the set with 3 and 2 as its only elements is the same set as the set of all elements chosen from I which are smaller than 4 but greater than 1. The identity of a set thus does not depend on the description or sequence of its elements, but only on the elements themselves.

To show that two sets are equal, e.g. that $A = B$, we have to show that they have the same elements, in other words we have to show that $(\forall x)(x \in A \Leftrightarrow x \in B)$.

Section 2: Notation for sets

- The "$\in$" symbol means "is an element of" while the "$\notin$" symbol means "is not an element of".
- The list of all the elements of a finite set is written between curly brackets {} with commas between them.

Examples:

(1) {2, 3} or {3, 2}

(2) {-11, 7, π}

(3) {1, 2, 3,...,17}, the set of integers starting with 1 and ending with 17 and including all those in between 1 and 17

- Although there is no last element, the list describing the pattern according to which the elements of an infinite set can be written down, is also written between curly brackets, while the "..." denotes the fact that the process is a continuing one.

Examples:

(1) {1, 2, 3, ...} is the set of all integers greater than or equal to 1

(2) {..., -3, -2, -1, 0, 1, 2, 3, ...} is the set of all integers, I.

- *Set builder notation* is a notation that expresses that we mean a set of all elements with a certain feature. The general notation is

  $\{x \mid x$ has the following features$\}$

  or briefly

  $\{x \mid P(x)\}$

- "x" can be any letter; the variables x, y, z are used most often
- "x" has to come from a previously given set, say X ; there are two ways in which to refer to this previously given set: $\{x \in X \mid x \ldots\}$ or $\{x \mid x \in X$ and $x \ldots\}$
- The X is sometimes omitted, but this happens only if it is clear which "X" is being referred to
- the mathematical description of the relevant features of "x" follows after the "|"
- sometimes ":" is used instead of "|".

  Examples:

  (1) $\{x \mid x$ is an integer and x is divisible by 3$\} = \{\ldots, -9, -6, -3, 0, 3, 6, 9, \ldots\}$

  (2) $\{x : x$ is a natural number and x is bigger than 3$\} = \{4, 5, 6, 7, \ldots\}$

- A set can also be denoted by a single letter, e.g. R, N, I, C, X , A, D and so on.
- The empty set (that is the set with no elements) is denoted by "Ø", which means {}. Remember that a set can be empty. Never simply presuppose when considering some set that it has elements, especially when the set builder notation is used to describe the set and no elements are explicitly listed. (A bag is a bag, even if it is an empty bag.)

*Examples***:**

(1) Ø $= \{ x \in$ R $\mid x \neq x \}$

There exists no object that is not equal to itself.

(2) Ø $= \{ x \in$ I $\mid x > 5$and $x < -11\}$

There is no x that satisfies both these conditions simultaneously**.**

Section 3: Subsets

Introduction

If all the elements of a set X are also elements of a set Y, we say that "X is a subset of Y". For example, all natural numbers (that is elements of N) are integers also (that is elements of I).

Thus N is a subset of I. In the same way I is a subset of the rational numbers, Q, and Q is a subset of the real numbers, R.

Notation

- $X \subseteq Y$: X is a subset of Y (or $Y \supseteq X$: Y contains X). If $X \subseteq Y$ and Y contains at least one element that is not an element of X (that is $Y \nsubseteq X$), we say that X is a so-called "proper" subset of Y and that Y strictly contains X. It is written $X \subset Y$ or $Y \supset X$. Sometimes the symbol "$\subsetneq$" is used for "is a subset of but is not equal to".

$X \subseteq Y \Leftrightarrow (\forall x)(x \in X \rightarrow x \in Y)$

$X \subset Y \Leftrightarrow (\forall x)(x \in X \rightarrow x \in Y) \wedge X \neq Y \Leftrightarrow X \subseteq Y \wedge Y \nsubseteq X$

The relation "$\subseteq$" is anti-symmetrical:

$(X \subseteq Y \wedge Y \subseteq X) \rightarrow X = Y$

The converse is of course also valid:

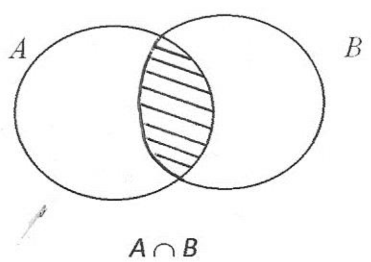
$X = Y \rightarrow (X \subseteq Y \wedge Y \subseteq X)$

- Empty sets: $\varnothing \subseteq X, \ \forall X$
- Reflexivity: $X \subseteq X, \ \forall X$
- Transitivity: $\forall X, Y, Z : (X \subseteq Y \wedge Y \subseteq Z) \rightarrow X \subseteq Z$.

Section 4: Operations on sets

4.1 Intersection

The intersection of two sets A and B is another set, namely the set consisting of all the elements which are elements of A and also elements of B. We write the intersection as $A \cap B$.



$A \cap B$

Formally (where ":=" means "is per definition equal to"):

$A \cap B := \{x \mid x \in A \ \wedge \ x \in B\}$

The operation "$\cap$" is commutative and associative because the connective "$\wedge$" has these features.

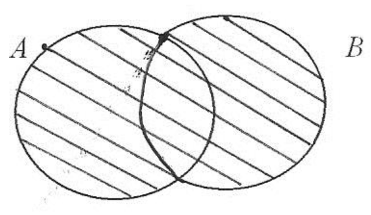
Example:

Show that intersection is commutative:

$A \cap B \ = \{x \mid x \in A \ \wedge \ x \in B\} \ = \ \{x \mid x \in B \ \wedge \ x \in A\} \ = \ B \cap A$

4.2 Union

The union of two sets *A* and *B* is another set, namely the set of all those elements that are either elements of *A* or elements of *B*. (Remember that we use the inclusive "or" in this context.) We write the union as $A \cup B$.



$A \cup B$

Formally

$A \cup B := \{x \mid x \in A \ \vee \ x \in B\}$

The operation "$\cup$" is commutative and associative because the connective "$\vee$" has these features.

Example:

Show that union is associative:

$$\begin{aligned}
x \in A \cup (B \cup C) \quad &\Leftrightarrow \quad x \in A \ \vee \ x \in (B \cup C) \\
&\Leftrightarrow \quad x \in A \ \vee \ (x \in B \ \vee \ x \in C) \\
&\Leftrightarrow \quad (x \in A \ \vee \ x \in B) \ \vee \ x \in C \\
&\Leftrightarrow \quad x \in (A \cup B) \ \vee \ x \in C \\
&\Leftrightarrow \quad x \in (A \cup B) \cup C
\end{aligned}$$

Thus $A \cup (B \cup C) \ = \ (A \cup B) \cup C$

(Remember $X = Y \ \Leftrightarrow \ \forall z(z \in X \ \Leftrightarrow \ z \in Y)$.)

4.3 Distributive rules

Two different distributive laws are valid for union and intersection combined.

(1) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(2) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof for (1):

For all x:

$x \in (A \cup (B \cap C)) \Leftrightarrow x \in A \lor x \in (B \cap C)$

$\Leftrightarrow x \in A \lor (x \in B \land x \in C)$

$\Leftrightarrow (x \in A \lor x \in B) \land (x \in A \lor x \in C)$ ("or" is distributive over "and")

$\Leftrightarrow (x \in A \cup B) \land (x \in A \cup C)$

$\Leftrightarrow x \in ((A \cup B) \cap (A \cup C))$

Thus $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

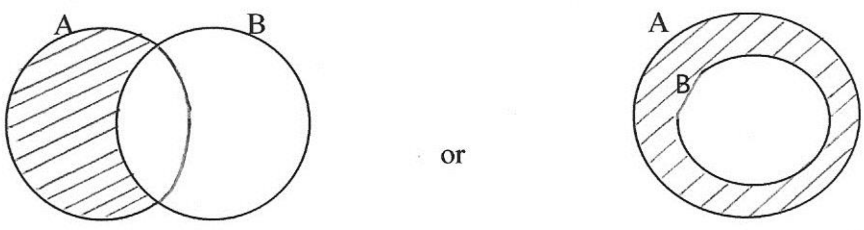
Now prove (2) above yourself.

4.4 Complement

If *A* and *B* are sets, then the relative complement of *B* with respect to *A*, written *A* - *B* (or *A**B*), is the set of all those elements of *A* that are not elements of *B*.

$A - B : \quad = \{x \mid x \in A \land \neg (x \in B)\}$

$\quad = \{x \mid x \in A \land x \notin B\}$



A – B A – B

Example:

The set of all irrational numbers is the relative complement of the set of rational numbers with respect to the set of real numbers.

Comment

Sometimes, if B is a subset of A (thus $B \subseteq A$), the A in $A - B$ is simply presupposed and then we speak only of the complement (B^c) of B. This may, however, be done only if no confusion about A is possible.

Example:

"The set of irrational numbers is the complement of the set of rational numbers".

De Morgan's laws for sets

(1) $A - (B \cap C) = (A - B) \cup (A - C)$

(2) $A - (B \cup C) = (A - B) \cap (A - C)$

(Compare these laws to De Morgan's laws for sentences in your textbook. Note the correspondence between A - ... and $\neg$, between $\cap$ and $\wedge$, and between $\cup$ and $\vee$.)

Proof for (1):

For all x:

$$x \in A - (B \cap C) \Leftrightarrow x \in A \wedge \neg (x \in (B \cap C))$$

$$\Leftrightarrow x \in A \wedge \neg (x \in B \wedge x \in C)$$

$$\Leftrightarrow x \in A \wedge [\neg (x \in B) \vee \neg (x \in C)]$$

De Morgan's law for connectives

$$\Leftrightarrow [x \in A \wedge \neg (x \in B)] \vee [(x \in A) \wedge \neg (x \in C)]$$

Distributive law for $\wedge$ over $\vee$

$$\Leftrightarrow [x \in (A - B)] \vee [x \in (A - C)]$$

$$\Leftrightarrow x \in (A - B) \cup (A - C)$$

Thus $A - (B \cap C) = (A - B) \cup (A - C)$

Now prove (2) above yourself.

Section 5: Cartesian product, relations and functions

5.1 Ordered pairs

If we want to refer to two elements a and b in the sequence a first and b second, we write these elements between round brackets:

$$(a, b)$$

and call it the ordered pair (a, b) with first coordinate a and second coordinate b. Two ordered pairs (a, b) and (c, d) will in this sense be equal if and only if $a = c$ and $b = d$.

Example:

In Geometry, the pair (x, y) indicates the x- and y-coordinates of a point. The pair (3, 5) does not indicate the same point as the pair (5, 3).

5.2 Cartesian product

The Cartesian product $A \times B$ (read A cross B) of two sets A and B is the set of all pairs (a, b) of which the first coordinate a belongs to A and the second coordinate b belongs to B. In other words,

$$A \times B := \{(a, b) \mid a \in A \ \wedge \ b \in B\}$$

Note that, if $A = B$, then we may write A^2 instead of $A \times A$. Similarly $A^3 = A \times A \times A = \{(x, y, z) \mid x, y, z \in A\}$ and, for $n \geq 1$, $A^n = \{(a_1, a_2, ..., a_n) \mid a_1, a_2, ..., a_n \in A\}$.

Example:

If $A = \{a, b, c\}$ and $B = \{2, 3\}$ then

$$A \times B = \{(a, 2), (b, 2), (c, 2), (a, 3), (b, 3), (c, 3)\}$$

$$B \times A = \{(2, a), (3, a), (2, b), (3, b), (2, c), (3, c)\}.$$

Note that $A \times B \neq B \times A$ in this example.

5.3 Relations

Let A and B be any two sets. A *relation* r from A to B is any subset of $A \times B$:

$$r \subseteq A \times B.$$

If r is a relation from A to B, the domain $D(r)$ of r is the subset of A which consists of all first coordinates of all pairs in r, and the range $R(r)$ of r is the set of all second coordinates of the pairs in r. That is

$D(r) := \{x \mid (\exists y)((x, y) \in r)\}$ and $R(r) := \{y \mid (\exists x)((x, y) \in r)\}$.

Example 1:

Suppose $X = \{-1, 0, 1, 3\}$. Of which ordered pairs do the following relations consist?

$r_1 = \{(x, y) \in X \times X \mid y = 3x\}$

and $r_2 = \{(x, y) \in X \times X \mid y = x^2\}$

First r_1:

$x = -1 \quad y = 3x = -3$

$x = 0 \quad y = 3x = 0$

$x = 1 \quad y = 3x = 3$

$x = 3 \quad y = 3x = 9$

Because $r_1 \subseteq X \times X$, the ordered pairs of which r_1 consists are only $r_1 = \{(0, 0), (1, 3)\}$.

Now r_2:

$x = -1 \quad y = x^2 = 1$

$x = 0 \quad y = x^2 = 0$

$x = 1 \quad y = x^2 = 1$

$x = 3 \quad y = x^2 = 9$

Because $r_2 \subseteq X \times X$, the ordered pairs of which r_2 consists are $r_2 = \{(-1, 1), (0, 0), (1, 1)\}$.

Example 2:

Give the domain and range of the relations r_1 and r_2 defined above.

$D(r_1) = \{0, 1\}$ and $R(r_1) = \{0, 3\}$

$D(r_2) = \{-1, 0, 1\}$ and $R(r_2) = \{0, 1\}$

5.4 Functions

Relations consisting of ordered pairs such that only one element in the range of these relations corresponds to every element in the domain of these relations, belong to a special type of relation.

Example:

$r = \{(x, y) \mid y = x + 5, (x, y) \in \mathrm{R}^2\}$

Let X and Y be two given sets. A *function* $f : X \to Y$ from X to Y is a set of ordered pairs (x, y) from $X \times Y$ (that is, a relation $f \subseteq X \times Y$) such that there exists for every $x \in X$ one and only one $y \in Y$ with $(x, y) \in f$.

A function $f : X \to Y$ is thus a relation from X to Y, because it is a subset of $X \times Y$, but not all relations are functions.

$f \subseteq X \times Y$ is a function from X to Y if and only if

- $D(f) = X$ and $R(f) \subseteq Y$ and
- if $(x, y_1) \in f$ and also $(x, y_2) \in f$, then $y_1 = y_2$, i.e. no two different pairs from f have the same first coordinates.

If f is a function and $(x, y) \in f$,we often write $y = f(x)$ and then we call y (or $f(x)$) the *function value* of f in x or the *image* of x under the function (or *mapping*) f . From the definition it is obvious that every function f maps every $x \in D(f)$ in a *unique way* onto some element of $R(f)$. We write $f : x \mapsto f(x)$. The writing of $y = f(x)$ leads us to talk of y being *a function of* x. In this context we call x the *independent variable* and y the *dependent variable*.

Examples:

Let $X = \{a, b, c, d, e\}$ and $Y = \{x, y, z\}$.

(1) Then $f = \{(a, x), (b, x), (c, y), (d, y), (e, y)\}$ is a function from X to Y.

(2) The relation $r = \{(a, x), (a, y), (b, y), (c, y), (d, z), (e, z)\}$ is, however, not a function from X to Y.

Summarising example:

Suppose we want to represent mathematically the binary relation "is married to". Let H be a set representing all married people and G be the subset of $H \times H$ that represents all married pairs. Then we have

$G = \{(x, y) \in H \times H \mid x \text{ is married to } y\}$.

(We are omitting polygamy.)

Thus, if $(a, b) \in G$, it means that the person indicated by a is married to the person indicated by b.

Note that $G : D(G) \to R(G)$ is a function: If $(x, y_1), (x, y_2) \in G$ then $y_1 = y_2$, because x is married to only one y.

CHAPTER 3 RECURSION AND INDUCTION

Section 1: Definitions by recursion

Introduction

Suppose we have a (non-empty) set A and a set of functions, F, all of which are functions of A^n (for some n) in A. Thus $f \in F$ is n-ary and we have f : $A^n \rightarrow A$, which means that if we choose any n elements $a_1, a_2, ..., a_n \in A$, then the function value $f(a_1, a_2, ..., a_n)$ is an element of A again. Here n can be any of the numbers 1, 2, 3, ...

Example:

Let A = N = {0, 1, 2, 3, ...} and $F = \{S\}$, where S is the unary function $S : A \rightarrow A$ that is defined as follows:

$S(n) = n + 1; \quad n \in$ N

("S" for "successor function").

Suppose now that G is a subset of A. We call an element of G a *generator* or *generating element*. If we apply functions from F to elements of G, we get elements of A (which do not necessarily have to be elements of G). We can apply this process repeatedly, also on elements that have already been generated in this way.

Example:

In our example above A = N, $F = \{S\}$, S unary. Then we can choose $G = \{0\}$ and generate the elements 0, $S(0)$, $S(S(0))$, $S(S(S(0)))$, . . ., namely the elements 0, 1, 2, ... In this instance we have the case that all natural numbers can be generated from the generating set G, that is, starting with the single element 0, by applying the function S repeatedly.

If we start with any G and F, we choose the name G^* for the set of all elements (of A) that we can generate by repeatedly applying functions from F.

Example:

Let A be the set of all possible strings of symbols that can be formed from the set of symbols

$S = \{x, +, \cdot,), (, 0, 1, 2, 3, ..., \}$.

An example of a string in A is

x++(3· (·x ·+

This string, however, does not make sense in terms of the usual meanings we associate with "+" and "·".

Suppose we want to describe the set of all well-formed strings, for example say (x + 3) ·5. We do this by viewing the set of well-formed expressions as the G^* that is generated by $G = \{x, 0, 1, 2, 3, ...\}$ and the set $F = \{f_+, f_\cdot\}$ of binary functions. The functions $f_+ : A \to A$ and $f_\cdot : A \to A$ are defined as follows: If s and t are any two strings (that are elements of A), then

$f_+ (s, t) = (s) + (t)$ and

$f_\cdot (s, t) = (s) \cdot (t)$

Examples of applications of these functions:

(1) $f_+ (++3, xx() = (++3) + (xx()$

(2) $f_+ (x + 3, 5) = (x + 3) + (5)$

(3) $f_\cdot (x + 3, 5) = (x + 3) \cdot (5)$

G^* is the set of all those strings in A (the set of all possible strings) that can be generated from G by applying f_+ and $f_\cdot$ any number of times in any sequence on any choice of starting elements from G (generators). An example of such a string is

(((3)·(x)) ·(8)) ·(((x)·(x))+(7))

G^* is thus the set of well-formed expressions we were looking for.

Example:

We now want to view the set of well-formed sentences of propositional logic as a G^* generated by a G and an F. To do this, we choose G (the generators) as the set of propositional symbols that we are going to use:

$G = \{p_0, p_1, p_2, ...\}$

Let H be the union of the set of all connectives and brackets and the set G, thus

$H = \{\neg, \wedge, \vee, \to, \leftrightarrow,), (\} \cup G.$

A is now defined as the set of all possible (finite) strings of symbols from H. Remember that a single symbol is a string with length one, in other words $H \subseteq A$. The following strings are for example elements of A:

$\neg \leftrightarrow p(($

$p \leftrightarrow q$

$(p) \leftrightarrow (q)$

We now proceed as follows:

Let $F = \{f_\neg, f_\wedge, f_\vee, f_\rightarrow, f_\leftrightarrow\}$

where $f_\neg : A \rightarrow A$ is a unary function and all the others are binary functions from $A \times A$ to A. The functions are defined as follows:

If $s, t \in A$ (in other words, if s and t are any two strings from our "alphabet") then

$f_\neg(s) = \neg(s)$

$f_\wedge(s, t) = (s) \wedge (t)$

$f_\vee(s, t) = (s) \vee (t)$

$f_\rightarrow(s, t) = (s) \rightarrow (t)$

$f_\leftrightarrow(s, t) = (s) \leftrightarrow (t)$

It is easy to see that the G^* that we now generate from the G and F, in other words all strings that are formed by the propositional letters p_0, p_1, p_2, ... and the connectives according to the above rules, are precisely the well-formed formulas of propositional logic as defined in your textbook.

1.1 Definitions of features (properties)

Suppose we consider a feature, say P (predicate symbol), of elements of A. We can define the feature in such a way that we ensure that precisely the elements of G^* will have this feature, that is for every element a of A:

a has P if and only if $a \in G^*$, that is $P(a) \Leftrightarrow a \in G^*$.

We do it as follows:

(1) We ensure that all elements of G has feature P.

(2) We ensure that if $f \in F$ is an n-ary function and if $a_1, a_2, a_3, ..., a_n$ $(\in A)$ have the feature P, then the result $f(a_1, a_2, a_3, ..., a_n)$ has feature P.

(3) We ensure that an element of A has the feature P only if it has it according to (1) or (2).

Example 1:

$A = \{0, 1, 2, ...\} = N$

$G = \{0\}$

$F = \{S\}$ (the successor function, $S(n) = n + 1$; $n \in N$).

Consider the feature "is a natural number".

(1) 0 is a natural number, thus all elements of G has the feature "is a natural number".

(2) If n is a natural number, then $S(n) = n + 1$ is one too, thus the feature P stays valid if we apply S.

(3) In this case then $G^* = N$ which is equal to the set of all elements with feature P.

(Note that it is not always the case that $G^* = A$, in other words that the whole A is generated by G.)

Example 2:

A is the set of all strings from the alphabet $H = \{\neg, \wedge, \vee, \rightarrow, \leftrightarrow,), (, p_0, p_1, p_2, ...\}$

$G = \{p_0, p_1, p_2, ...\}$

$F = \{f_\neg, f_\wedge, f_\vee, f_\rightarrow, f_\leftrightarrow\}$

Let the feature P be "is a well-formed sentence". Then we can specify that this feature be defined by the following description (conditions):

(1) Every element of G is a well-formed sentence, in other words every p_i has the feature P.

(2) If θ and ψ are well-formed sentences (in other words if they have feature P), then

$f_\neg(\theta) = \neg(\theta)$

$f_\wedge(\theta, \psi) = (\theta) \wedge (\psi)$

$f_\vee(\theta, \psi) = (\theta) \vee (\psi)$

$f_\rightarrow(\theta, \psi) = (\theta) \rightarrow (\psi)$

$f_\leftrightarrow(\theta, \psi) = (\theta) \leftrightarrow (\psi)$

are well-formed sentences too (all have feature P).

(3) Only strings that qualify according to (1) and (2) are well-formed (have feature P).

(1), (2) and (3) define the feature P, and the set of strings that have feature P is precisely G^*, the set of well-formed sentences. We say that we have defined P, or G^*, *by recursion*.

1.2 Definitions of functions

Say we want to define by recursion a function on the domain G^*. We can define a function $h : G^* \to U$ by specifying the following:

(1) for every $g \in G$ we say what the $h(g)$ ($\in U$) is

and

(2) for every n-ary $f \in F$ and every choice of n elements $a_1, a_2, a_3, ..., a_n$ from G^*, for which we already know what the $h(a_1), h(a_2), h(a_3), ..., h(a_n)$ ($\in U$) are, we specify what the $h(f(a_1, a_2, a_3, ..., a_n))$ is in terms of $h(a_1), h(a_2), h(a_3), ..., h(a_n)$.

(Step (2) is usually carried out by specifying a function k such that

$$h(f(a_1, a_2, a_3, ..., a_n)) = k\,(h(a_1), h(a_2), h(a_3), ..., h(a_n)).)$$

Example 1:

Given $A = \mathrm{N}$

$G = \{0\}$

$F = \{S\}$ where $S(n) = n + 1$,

define by recursion on the natural numbers, the function "add 5". We call the function h.

(1) We have to define h on the elements of G : $h(0) = 0 + 5 = 5$

(2) If we already have $h(n)$, we have to say how to get $h(S(n))$. We do it as follows:

$h(S(n)) = h(n) + 1$

Thus $h(0) = 5$

$h(1) = h(0) + 1$

$= 5 + 1$

$= 6$

$= 1 + 5$

$h(2) \quad = h(1) + 1$

$= 6 + 1$

$= 7$

$= 2 + 5$

$h(n) \quad = n + 5, \quad \forall n \in N.$

Example 2:

We want to define a function h that gives to every well-formed sentence θ (of propositional logic) its length as $h(\theta)$, where by “length” of sentence θ we mean the number of connectives that appear in θ (counted more than once if the connective appears more than once).

(1) $h(p_i) = 0, \ \forall p_i \in G$

(2) $h(f_\neg(\theta)) = h(\neg(\theta)) = h(\theta) + 1$

$h(f_\wedge(\theta, \psi)) = h((\theta) \wedge (\psi)) = h(\theta) + h(\psi) + 1$

$h(f_\vee(\theta, \psi)) = h((\theta) \vee (\psi)) = h(\theta) + h(\psi) + 1$

$h(f_\rightarrow(\theta, \psi)) = h((\theta) \rightarrow (\psi)) = h(\theta) + h(\psi) + 1$

$h(f_\leftrightarrow(\theta, \psi)) = h((\theta) \leftrightarrow (\psi)) = h(\theta) + h(\psi) + 1$

Then the function h gives to each well-formed sentence its length and that is precisely the number of occurrences of connectives in the sentence.

Example 3:

We want to define a function h that assigns to every well-formed sentence (of propositional logic) a truth value, T or F, when we begin with a specific assignment of truth values to the atomic sentences p_i. So we choose G to be the set of atomic sentences: $G = \{p_0, p_1, p_2, ...\}$, while G^* is the set of all well-formed sentences as before (see the example above showing the set of all well-formed sentences of propositional logic as a G^* generated by a G and an F). We now specify the function $h : G^* \rightarrow \{T, F\}$ as follows:

(1) For every $p_i \in G$ we specify a truth value $h(p_i)$, which is either T or F.

(2) For every complex sentence we must specify what its truth value is, given the truth value(s) of its components. The complex sentence can have one of 5 forms:

$$h(f_{\neg}(\theta)) = h(\neg(\theta)) = \begin{cases} F \ if \ h(\theta) = T \\ T \ if \ h(\theta) = F \end{cases}$$

$$h(f_{\wedge}(\theta, \psi)) = h((\theta) \wedge (\psi)) = \begin{cases} F \ if \ h(\theta) = F \ or \ h(\psi) = F \\ T \ if \ h(\theta) = T \ and \ h(\psi) = T \end{cases}$$

$$h(f_{\vee}(\theta, \psi)) = h((\theta) \vee (\psi)) = \begin{cases} F \ if \ h(\theta) = F \ and \ h(\psi) = F \\ T \ if \ h(\theta) = T \ or \ h(\psi) = T \end{cases}$$

$$h(f_{\rightarrow}(\theta, \psi)) = h((\theta) \rightarrow (\psi)) = \begin{cases} F \ if \ h(\theta) = T \ and \ h(\psi) = F \\ T \ if \ h(\theta) = F \ or \ h(\psi) = T \end{cases}$$

$$h(f_{\leftrightarrow}(\theta, \psi)) = h((\theta) \leftrightarrow (\psi)) = \begin{cases} F \ if \ h(\theta) \neq h(\psi) \\ T \ if \ h(\theta) = h(\psi) \end{cases}$$

The function h which is defined in this way now assigns a truth value to every well-formed sentence.

Section 2: Proofs by induction

We have A, F, G and G^* as before. Consider some feature, let's call it P again, that elements of A may or may not have. A proof by induction will follow the following scheme:

Premises

(1) All the elements of G have feature P. (This may be self-evident, or it may need a proof. Whatever the case, such a premise has to be justified.)

(2) The feature P is hereditary, that is, it stays valid when functions from F are applied to elements in A (or in G^*) already having feature P. In other words, if $f \in F$ is an n-ary function, then if $a_1, a_2, a_3, ..., a_n \in$ A all have feature P, $f(a_1, a_2, a_3, ..., a_n)$ also has this feature. (This premise usually needs a proof.)

Conclusion

All the elements of G^* have feature P.

Justification for the above induction scheme

Because all the elements in G^* are built up by repeatedly applying functions from F to the generators from G, they all have to have feature P: the starting elements from G have P and P remains valid in each step in the build-up of an element in G^*.

Example 1:

$A =$ N

$G = \{0\}$

$F = \{S\}$, the successor function where $S(n) = n + 1$

$G^* =$ N (specifically here - remember that it is not always the case that $G^* = A$).

Let P be a feature that natural numbers may have. Then the induction scheme states that

(1) if 0 (the only element in G) has feature P

and

(2) if feature P remains valid under the application of S (the only element in F) to a natural number n having P as feature, that is, $(\forall n)(P(n) \rightarrow P(S(n))$, then all natural numbers (that is elements in G*) have feature P, in other words we can conclude that $(\forall n)P(n)$.

Example of the above for a given feature P

Let $P(n)$ be the following feature that some natural number n may have: "the sum of the first n+1 natural numbers (in other words from zero to n) is $\frac{1}{2}n(n+1)$"; thus $P(n)$: $0 + 1 + 2 + 3 + 4 + ... + n = \frac{1}{2} n (n+1)$.

We want to prove by induction that all natural numbers have feature P, that is that $(\forall n)P(n)$.

The induction scheme states then that we have to prove the following two premises to finish this proof:

(1) 0 has feature P, i.e. $P(0)$ and

(2) if any n has feature P, then $n + 1$ has this feature, i.e. $(\forall n)(P(n) \rightarrow P(n + 1))$.

Proof:

(1) What does $P(0)$ mean? $P(0)$: $0 = \frac{1}{2}\, 0\, (0+1)$, which is obviously true.

$P(0)$ has been proved.

(2) Suppose that $P(n)$ is valid. In other words suppose

$$0 + 1 + 2 + 3 + 4 + ... + n = \frac{1}{2} n (n+1).$$

We want to prove that P(n + 1) is valid, in other words we want to prove that

$$0 + 1 + 2 + 3 + 4 + ... + n + (n + 1) = \frac{1}{2}(n+1)((n+1) + 1).$$

We can do it as follows:

$$\begin{aligned} 0 + 1 + 2 + 3 + 4 + ... + n + (n + 1) &= (0 + 1 + 2 + 3 + 4 + ... + n) + (n + 1) \\ &= \frac{1}{2} n (n+1) + (n + 1) \\ &= \frac{1}{2}(n+1) n + \frac{1}{2}(n+1) 2 \\ &= \frac{1}{2}(n+1)(n + 2) \\ &= \frac{1}{2}(n+1)((n+1) + 1) \end{aligned}$$

Then $P(n + 1)$ is true and we have finished.

Example 2:

Let A, G and F be as defined for the previous examples from propositional logic, and let G^* be the set of all well-formed sentences. We want to prove (by induction) that all well-formed sentences (that is all $s \in G^*$) have the following feature $P(s)$:

"s is logically equivalent to a well-formed sentence containing $\wedge$ and $\neg$ as its only connectives" (in other words logically equivalent to a sentence that does not contain $\vee$, $\rightarrow$ or $\leftrightarrow$ in any way).

We now claim that the induction scheme states that we can prove $(\forall s)P(s)$ by proving two things:

(1) all the elements of G (the generators p_i) have feature P

(2) feature P remains valid under application of all the elements of $F = \{f_\neg, f_\wedge, f_\vee, f_\rightarrow, f_\leftrightarrow\}$ on all well-formed sentences (that is all elements of G^*).

Proof:

(We leave out some brackets in the proof for easier reading.)

(1) It is evident that a generator p_i contains none of $\vee$ or $\rightarrow$ or $\leftrightarrow$ and that $p_i = p_i$.

(2) We consider the hereditability of the feature P under application of each element of F separately:

(i) $f_\neg$: Suppose sentence s has feature P. In other words $s \equiv s'$, where s' contains none of the connectives $\vee$ or $\rightarrow$ or $\leftrightarrow$. Then

$f_\neg(s) = \neg(s) \equiv \neg(s')$

and $\neg(s')$ contains none of $\vee$ or $\rightarrow$ or $\leftrightarrow$. Thus $f_\neg(s)$ has feature P.

(ii) $f_\wedge$: Suppose sentences s_1 and s_2 have feature P. In other words $s_1 \equiv s_1'$ and $s_2 \equiv s_2'$, where neither s_1' nor s_2' contains any of the connectives $\vee$ or $\rightarrow$ or $\leftrightarrow$. Then

$f_\wedge(s_1, s_2) = (s_1) \wedge (s_2) \equiv (s_1') \wedge (s_2')$

which contains none of $\vee$ or $\rightarrow$ or $\leftrightarrow$. Thus $f_\wedge(s_1, s_2)$ has feature P.

(iii) $f_\vee$: Suppose sentences s_1 and s_2 have feature P. In other words $s_1 \equiv s_1'$ and $s_2 \equiv s_2'$, where neither s_1' nor s_2' contains any of the connectives $\vee$ or $\rightarrow$ or $\leftrightarrow$. Then

$f_\vee(s_1, s_2) = (s_1) \vee (s_2)$

$\equiv \neg(\neg s_1 \wedge \neg s_2)$ (De Morgan)

$\equiv \neg(\neg s_1' \wedge \neg s_2')$

which contains none of $\vee$ or $\rightarrow$ or $\leftrightarrow$. Thus $f_\vee(s_1, s_2)$ has feature P.

(iv) $f_\rightarrow$: Suppose sentences s_1 and s_2 have feature P. In other words $s_1 \equiv s_1'$ and $s_2 \equiv s_2'$, where neither s_1' nor s_2' contains any of the connectives $\vee$ or $\rightarrow$ or $\leftrightarrow$. Then

$f_\rightarrow(s_1, s_2) = (s_1) \rightarrow (s_2)$

$\equiv \neg s_1 \vee s_2$ (see 7.1, page 183 of the textbook for COS2661

or top of page 39 of the textbook for COS3761)

$\equiv \neg(s_1 \wedge \neg s_2)$ (De Morgan)

$\equiv \neg(s_1' \wedge \neg s_2')$

which contains none of $\vee$ or $\rightarrow$ or $\leftrightarrow$. Thus $f_{\rightarrow}(s_1, s_2)$ has feature P.

(v) $f_{\leftrightarrow}$: Suppose sentences s_1 and s_2 have feature P. In other words $s_1 \equiv s_1'$ and $s_2 \equiv s_2'$, where neither s_1' nor s_2' contains any of the connectives $\vee$ or $\rightarrow$ or $\leftrightarrow$. Then

$f_{\leftrightarrow}(s_1, s_2) = (s_1) \leftrightarrow (s_2)$

$\equiv (s_1 \rightarrow s_2) \wedge (s_2 \rightarrow s_1)$ (see 7.3, page 183 of the textbook for COS2661)

$\equiv (\neg s_1 \vee s_2) \wedge (\neg s_2 \vee s_1)$ (see 7.1, page 183 of the textbook for COS2661)

or top of page 39 of the textbook for COS3761)

$\equiv \neg (s_1 \wedge \neg s_2) \wedge \neg (s_2 \wedge \neg s_1)$ (De Morgan)

$\equiv \neg (s_1' \wedge \neg s_2') \wedge \neg (s_2' \wedge \neg s_1')$

which contains none of $\vee$ or $\rightarrow$ or $\leftrightarrow$. Thus $f_{\leftrightarrow}(s_1, s_2)$ has feature P.

This concludes the proof.

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