Foundations of Computer Science

Second Edition

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Chapter 2
Number Systems

Outlines

- Introduction
- Positional Number Systems
  – Base 10, 2, 8, 16.
- Nonpositional Number Systems
  – Roman Numerals

Objectives

After studying this chapter, the student should understand:

- The concept of number systems.
- Non-positional and positional number systems.
- Decimal, Binary, Hexadecimal and Octal system.
- Convert a number among binary, octal, hexadecimal, and decimal systems.
- Find the number of digits needed in each system to represent a particular value.

1-1 Introduction

The Definition of Number System

- A number can be represented using distinct symbols and differently in different systems.
- For example,
  - The two numbers $(2A)_{16}$ and $(52)_8$ both refer to the same quantity, $(42)_{10}$, but their representations are different
- Two groups
  - positional and non-positional systems

1-2 Position Number Systems

Overview

- In a **positional number system**, a number represented as:

$$\pm (S_{k-1} \ldots S_2\, S_1\, S_0 .\, S_{-1}\, S_{-2} \ldots S_{-l})_b$$

has the value of:

$$n = \pm\; S_{k-1} \times b^{k-1} + \ldots + S_1 \times b^1 + S_0 \times b^0 \;+\; S_{-1} \times b^{-1} + S_{-2} \times b^{-2} + \ldots + S_{-l} \times b^{-l}$$

in which S is the set of symbols, b is the **base** (or **radix**).

The Base includes Base10(Decimal), Base2(Binary), Base 16(Hexadecimal), or Base8(octal)。

The decimal system (base 10)

- The word decimal is derived from the Latin root **decem** (ten).
  - **base b = 10,** and
  - ten symbols: **S = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}**
- The symbols in this system are often referred to as decimal digits or just digits.
  - Integer examples
  - Real examples

Base 10 – Integers (1)

$$N = \pm \quad S_{k-1} \times 10^{k-1} + S_{k-2} \times 10^{k-2} + \ldots + S_2 \times 10^2 + S_1 \times 10^1 + S_0 \times 10^0$$

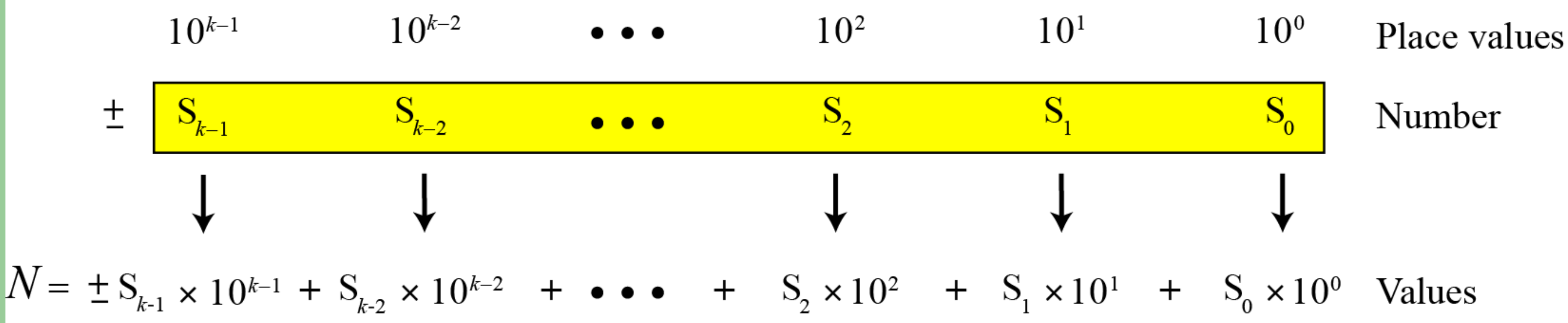


Figure 2.1 **Place values for an integer in the decimal system**

Base 10 – Integers (2)

- Example 2.1 shows the place values for the integer +224 in the decimal system

			10^2		10^1		10^0	Place values
			2		2		4	Number
N	=	+	2×10^2	+	2×10^1	+	4×10^0	Values

- Example 2.2 shows the place values for the decimal number −7508

			1000		100		10		1		Place values
			7		5		0		8		Number
N	=	−(	7×1000	+	5×100	+	0×10	+	8×1	)	**Values**

Base 10 – Reals

- A real – a number with a fractional part

	Integral part		Fractional part
$R = \pm$	$S_{k-1} \times 10^{k-1} + \ldots + S_1 \times 10^1 + S_0 \times 10^0$	+	$S_{-1} \times 10^{-1} + \ldots + S_{-l} \times 10^{-l}$

- Example 2.3 shows the place values for the real number +24.13.

	10^1		10^0		10^{-1}		10^{-2}	Place values
	2		4	•	1		3	Number
$R = +$	2×10	+	4×1	+	1×0.1	+	3×0.01	Values

The binary system (base 2)

- The word binary is derived from the Latin root **bini** (or two by two).
  - **base b = 2,** and
  - two symbols, **S = {0, 1}**
- The symbols in this system are often referred to as binary digits or bits (binary digit).
  - Integer examples
  - Real examples

Base 2 – Integers (1)

- We can represent an Integer as:

$$N = \pm\ S_{k-1} \times 2^{k-1} + S_{k-2} \times 2^{k-2} + \ldots + S_2 \times 2^2 + S_1 \times 2^1 + S_0 \times 2^0$$

	2^{k-1}	2^{k-2}	$\bullet\bullet\bullet$	2^2	2^1	2^0	Place values
$\pm$	S_{k-1}	S_{k-2}	$\bullet\bullet\bullet$	S_2	S_1	S_0	Number

$$N = \pm\ S_{k-1} \times 2^{k-1} + S_{k-2} \times 2^{k-2} + \bullet\bullet\bullet + S_2 \times 2^2 + S_1 \times 2^1 + S_0 \times 2^0 \quad \text{Values}$$

Figure 2.2 **Place values for an integer in the binary system**

Base 2 – Integers (2)

- Example 2.4 shows that the number $(11001)_2$ in binary is the same as 25 in decimal. The subscript 2 shows that the base is 2.

		2^4		2^3		2^2		2^1		2^0	Place values
		1		1		0		0		1	Number
N	=	1×2^4	+	1×2^3	+	0×2^2	+	0×2^1	+	1×2^0	Decimal

The equivalent decimal number is N = 16 + 8 + 0 + 0 + 1 = 25.

Base 2 – Reals (1)

	Integral part	.	Fractional part
$R = \pm$	$S_{k-1} \times 2^{k-1} + \ldots + S_1 \times 2^1 + S_0 \times 2^0$	$+$	$S_{-1} \times 2^{-1} + \ldots + S_{-l} \times 2^{-l}$

- Example 2.5 shows that the number $(101.11)_2$ in binary is equal to the number 5.75 in decimal.

		2^2		2^1		2^0		2^{-1}		2^{-2}	Place values
		1		0		1	•	1		1	Number
R	=	1×2^2	+	0×2^1	+	1×2^0	+	1×2^{-1}	+	1×2^{-2}	Values

The equivalent decimal number is R = 4+0+1+0.5+0.25 = 5.75.

The hexadecimal system (base 16)

- The word hexadecimal is derived from the Greek root **hex** (six) and the Latin root **decem** (ten).
  - **base b = 16**, and
  - sixteen symbols,

    S = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F}
- Symbol A, B, C, D, E, F are equivalent to 10, 11, 12, 13, 14, and 15 respectively.
- The symbols in this system are often referred to as hexadecimal digits.

Base 16 – Integers (1)

- We can represent an Integer as:

$$N = \pm S_{k-1} \times 16^{k-1} + S_{k-2} \times 16^{k-2} + \ldots + S_2 \times 16^2 + S_1 \times 16^1 + S_0 \times 16^0$$

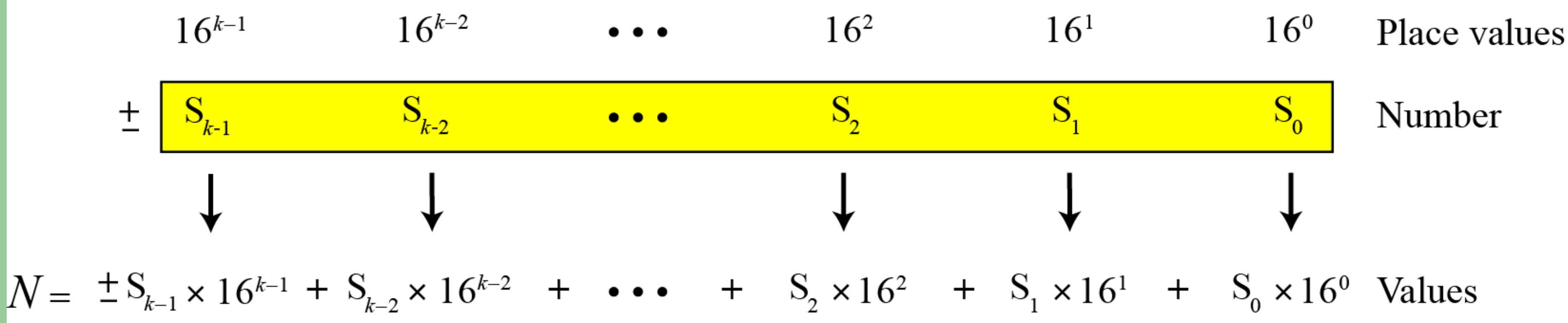


Figure 2.3 **Place values for an integer in the hexadecimal system**

Base 16 – Integers (2)

- Example 2.6 shows that the number $(2AE)_{16}$ in hexadecimal

		16^2		16^1		16^0	Place values
		2		A		E	Number
N	=	2×16^2	+	10×16^1	+	14×16^0	Values

The equivalent decimal number is $N = 512 + 160 + 14 = 686$.

The octal system (base 8)

- The word octal is derived from the Latin root **octo** (eight).
  - **base b = 8**, and
  - Eight symbols, **S = {0, 1, 2, 3, 4, 5, 6, 7}**
- Place values for an integer in the octal system

Base 8 – Integers (1)

- We can represent an Integer as:

$$N = \pm S_{k-1} \times 8^{k-1} + S_{k-2} \times 8^{k-2} + \ldots + S_2 \times 8^2 + S_1 \times 8^1 + S_0 \times 8^0$$

	8^{k-1}	8^{k-2}	$\cdots$	8^2	8^1	8^0	Place values
$\pm$	S_{k-1}	S_{k-2}	$\cdots$	S_2	S_1	S_0	Number
	↓	↓		↓	↓	↓	
$N =$	$\pm S_{k-1} \times 8^{k-1}$	$+ S_{k-2} \times 8^{k-2}$	$+ \cdots +$	$S_2 \times 8^2$	$+ S_1 \times 8^1$	$+ S_0 \times 8^0$	Values

Figure 2.4 **Place values for an integer in the octal system**

Base 8 – Integers (2)

- Example 2.7 shows that the number $(1256)_8$ in octal is the same as 686 in decimal.

		8^3		8^2		8^1		8^0	Place values
		1		2		5		6	Number
N	=	1×8^3	+	2×8^2	+	5×8^1	+	6×8^0	Values

The decimal number is N = 512 + 128 + 40 + 6 = 686.

Summary of the Base 10/2/8/16 positional systems (1)

Table 2.1 Summary of the four positional number systems

System	*Base*	*Symbols*	*Examples*
Decimal	10	0, 1, 2, 3, 4, 5, 6, 7, 8, 9	2345.56
Binary	2	0, 1	$(1001.11)_2$
Octal	8	0, 1, 2, 3, 4, 5, 6, 7	$(156.23)_8$
Hexadecimal	16	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F	$(A2C.A1)_{16}$

Summary (2)

The number 0 to 15 is represented in different systems

Table 2.2 Comparison of numbers in the four systems

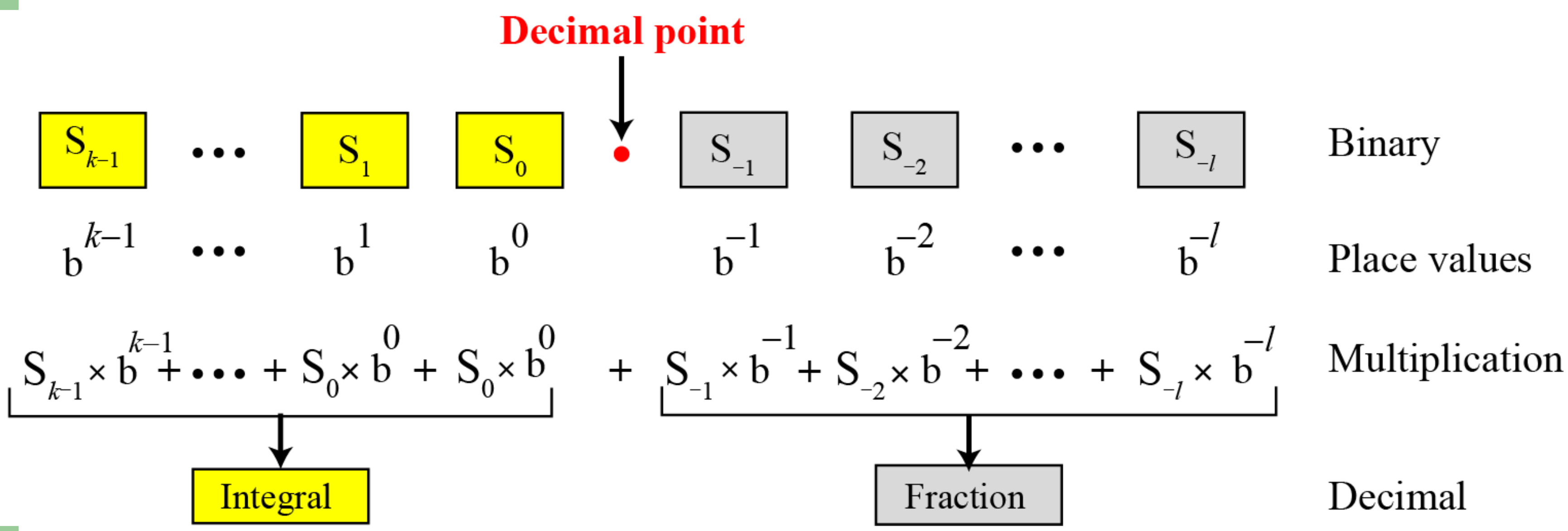
Decimal	*Binary*	*Octal*	*Hexadecimal*
0	0	0	0
1	1	1	1
2	10	2	2
3	11	3	3
4	100	4	4
5	101	5	5
6	110	6	6
7	111	7	7
8	1000	10	8
9	1001	11	9
10	1010	12	A
11	1011	13	B
12	1100	14	C
13	1101	15	D
14	1110	16	E
15	1111	17	F

Conversion

- The decimal system is more familiar than the other systems
- To convert a number in one system to the equivalent number in another system
  - Any base – Decimal
  - Binary – Hexadecimal
  - Binary – Octal
  - Octal – Hexadecimal

Any base to decimal conversion (1)

- Converting other bases to decimal (Fig. 2.5)



Any base to decimal conversion (2)

- Example 2.8 shows how to convert the binary number $(110.11)_2$ to decimal: $(110.11)_2$ = 6.75.

Binary	1		1		0	•	1		1
Place values	2^2		2^1		2^0		2^{-1}		2^{-2}
Partial results	4	+	2	+	0	+	0.5	+	0.25
Decimal: 6.75									

Any base to decimal conversion (3)

- Example 2.9 shows how to convert the hexadecimal number $(1A.23)_{16}$ to decimal.

Hexadecimal	1		A	•	2		3
Place values	16^1		16^0		16^{-1}		16^{-2}
Partial result	16	+	10	+	0.125	+	0.012

Decimal: 26.137

The result in the decimal notation is not exact, because $3 \times 16^{-2} = 0.01171875$. We have rounded this value to three digits (0.012).

Any base to decimal conversion (4)

- Example 2.10 shows how to convert $(23.17)_8$ to decimal.

Octal	2		3	•	1		7
Place values	8^1		8^0		8^{-1}		8^{-2}
Partial result	16	+	3	+	0.125	+	0.109

Decimal: 19.234

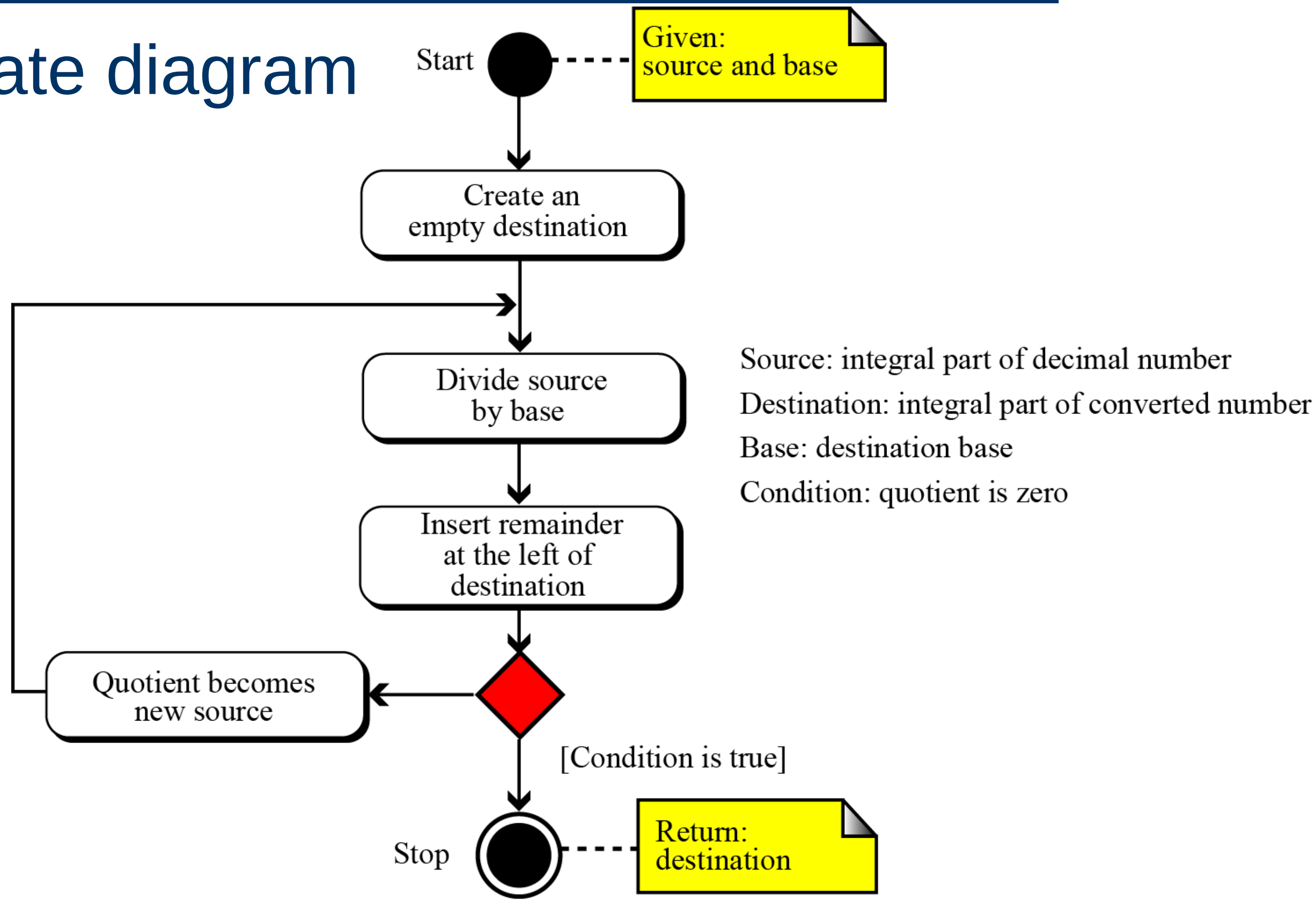
This means that $(23.17)8 \approx 19.234$ in decimal. Again, we have rounded up $7 \times 8^{-2} = 0.109375$.

Decimal to any base conversion

- Two procedures for converting a decimal number to its equivalent in any base.
  - Converting the integral part
  - Converting the fractional part

Converting the integral part (1)

UML's state diagram



Converting the integral part (2)

- The Figure shows the destination is made with each repetition. (process manually)

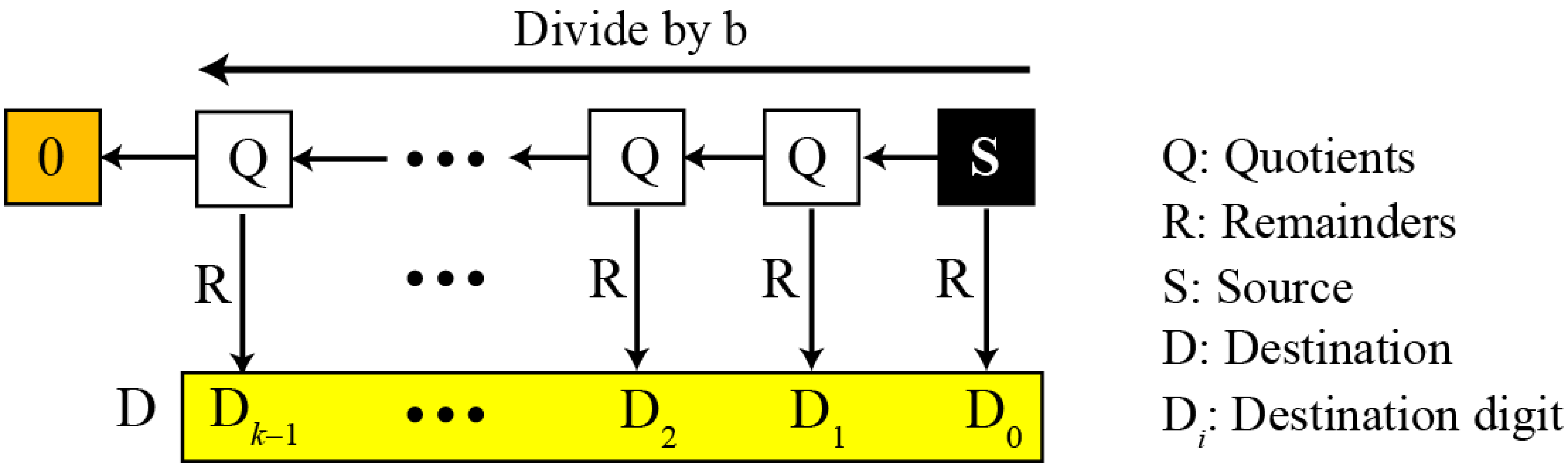
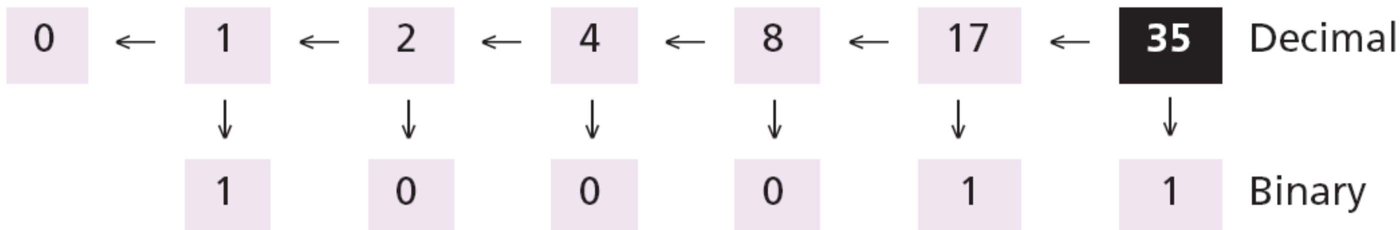


Figure 2.7

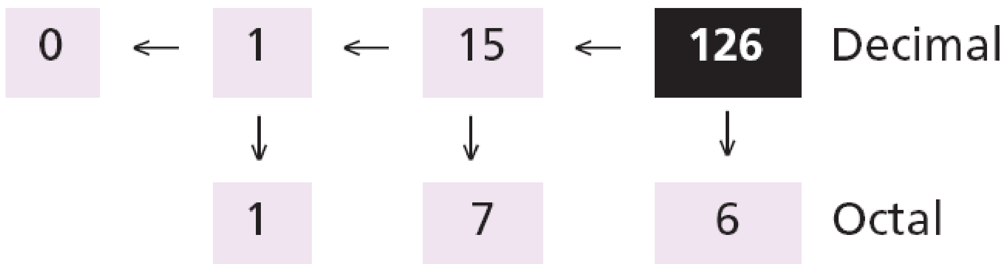
Converting the integral part of a number in decimal to other bases

Converting the integral part (3)

- Example 2.11 shows how to convert 35 in decimal to binary. The result is $35 = (100011)_2$.

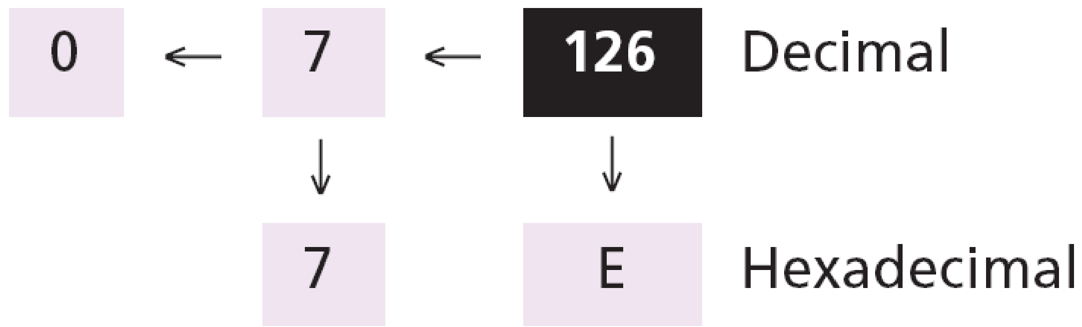


- Example 2.12 shows how to convert 126 in decimal to octal. The result is $126 = (176)_8$.



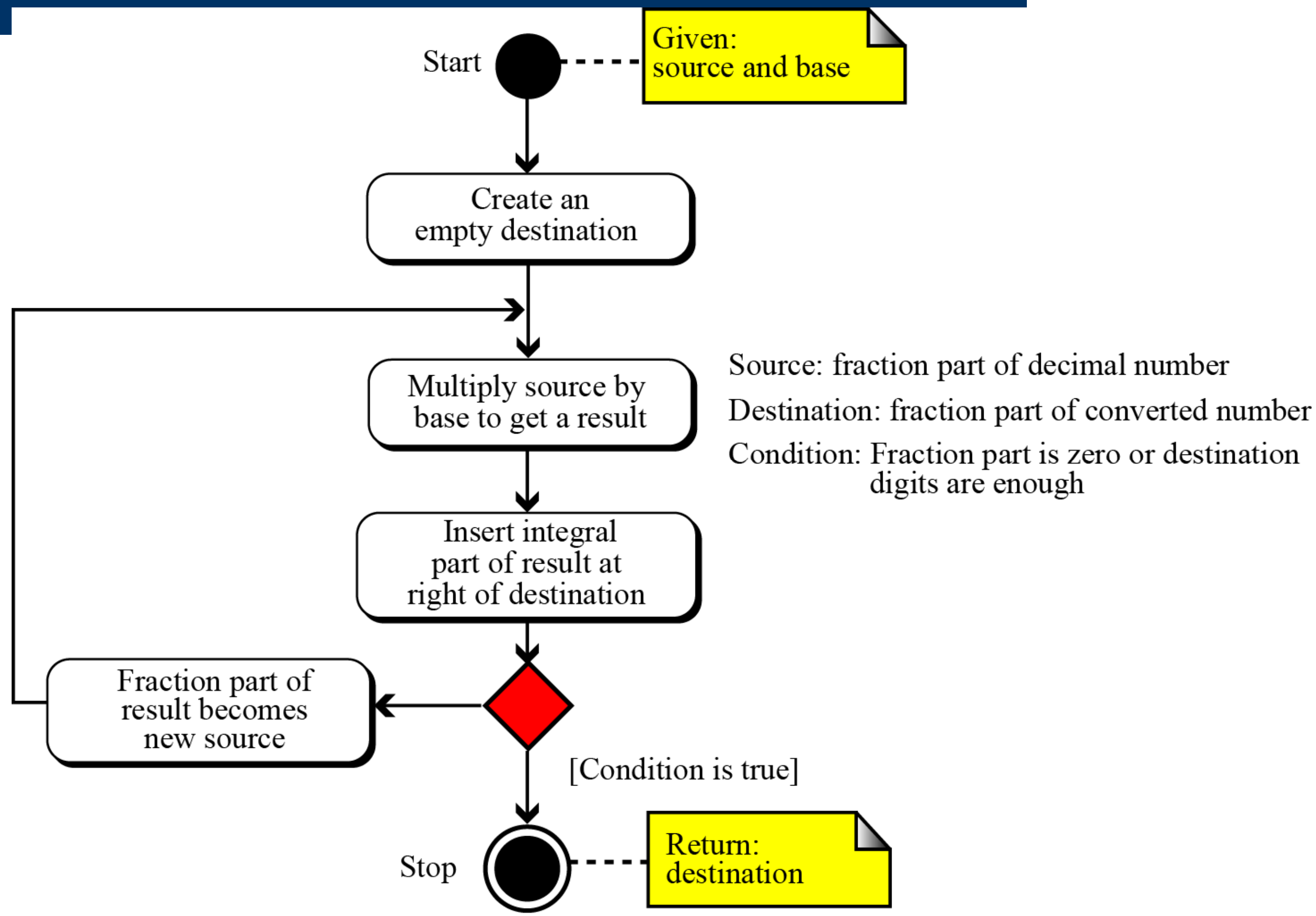
Converting the integral part (4)

- Example 2.13 shows how to convert 126 in decimal to hexadecimal. The result is 126 = $(7E)_{16}$



Converting the fractional part (1)

UML's State diagram



Converting the fractional part (2)

- The Figure shows the destination is made with each repetition. (process manually)

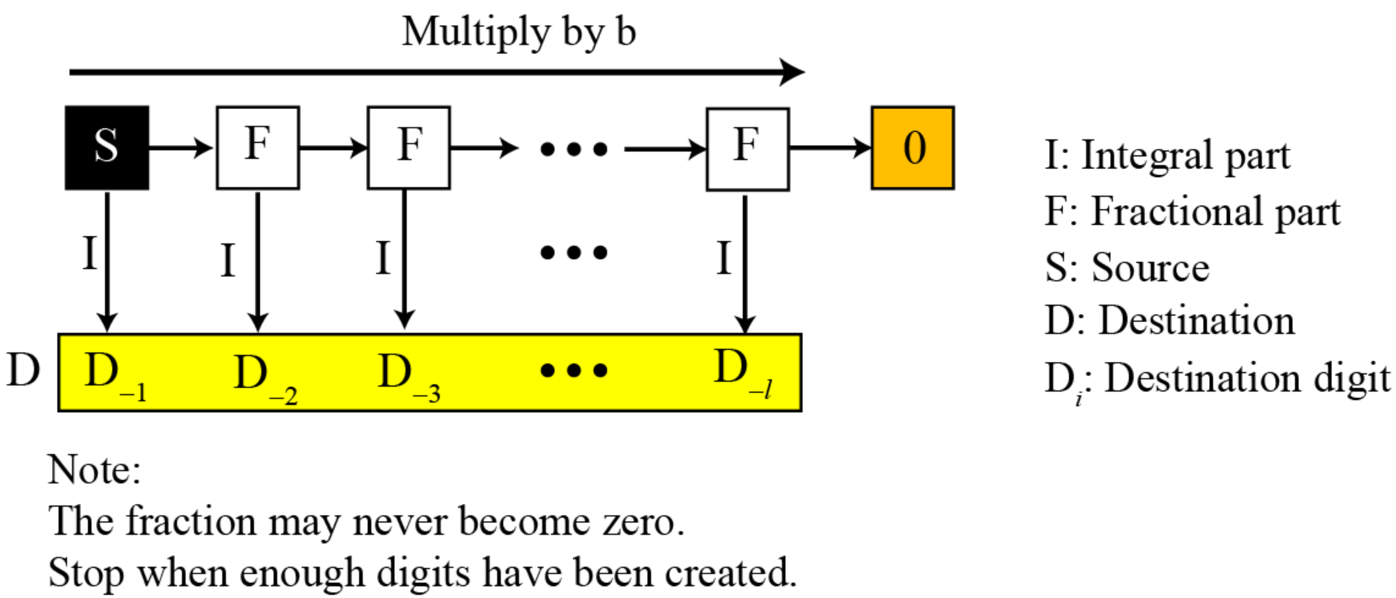
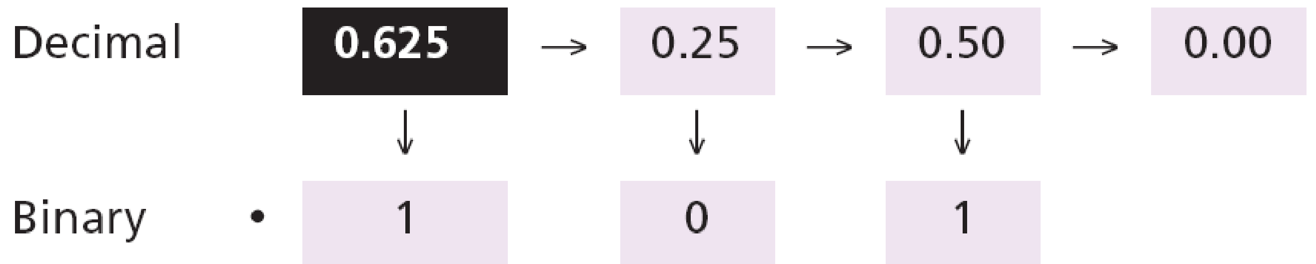


Figure 2.9

Converting the fractional part of a number in decimal to other bases

Converting the fractional part (3)

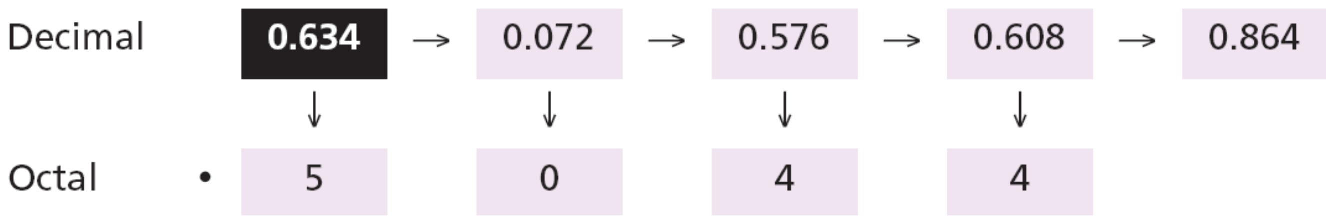
- Example 2.14 converts the decimal number 0.625 to binary.



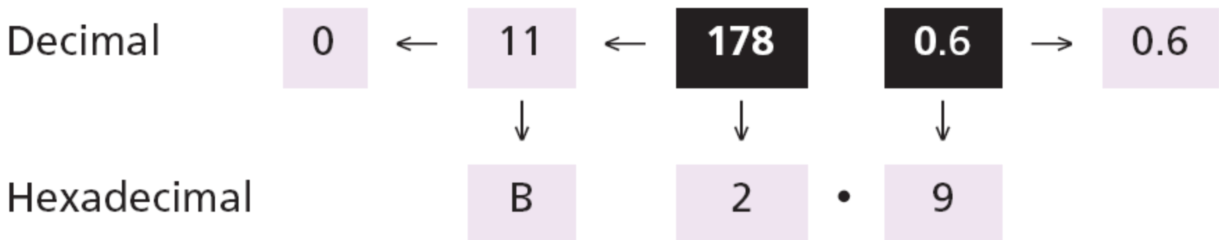
Since the number $0.625 = (0.101)_2$ has no integral part, the example shows how the fractional part is calculated.

Converting the fractional part (4)

- Example 2.15 shows how to convert 0.634 to octal using a maximum of four digits. The result is $0.634 = (0.5044)_8$.



- Example 2.16 shows how to convert 178.6 in decimal to hexadecimal using only one digit to the right of the decimal point. The result is $178.6 = (B2.9)_{16}$



Converting the fractional part (5)

- An alternative method for converting a small decimal integer (< 256) to binary is to break the number as the sum of numbers that are equivalent to the binary place values shown:

Place values	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
Decimal equivalent	128	64	32	16	8	4	2	1

For Example:

Decimal 165 =	128	+	0	+	32	+	0	+	0	+	4	+	0	+	1
Binary	1		0		1		0		0		1		0		1

Converting the fractional part (6)

- A method can be used to convert a decimal fraction to binary when the denominator is a power of two:
- Convert 27/64 to binary: The answer is $(0.011011)_2$

Place values	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}
Decimal equivalent	1/2	1/4	1/8	1/16	1/32	1/64	1/128

Decimal = 27/64	16/64	+	8/64	+	2/64	+	1/64
	1/4	+	1/8	+	1/32	+	1/64

Decimal 27/64 =	0	+	1/4	+	1/8	+	0	+	1/32	+	1/64
Binary	0		1		1		0		1		1

Binary-hexadecimal conversion (1)

- A relationship between the two bases: four bits in binary is one digit in hexadecimal

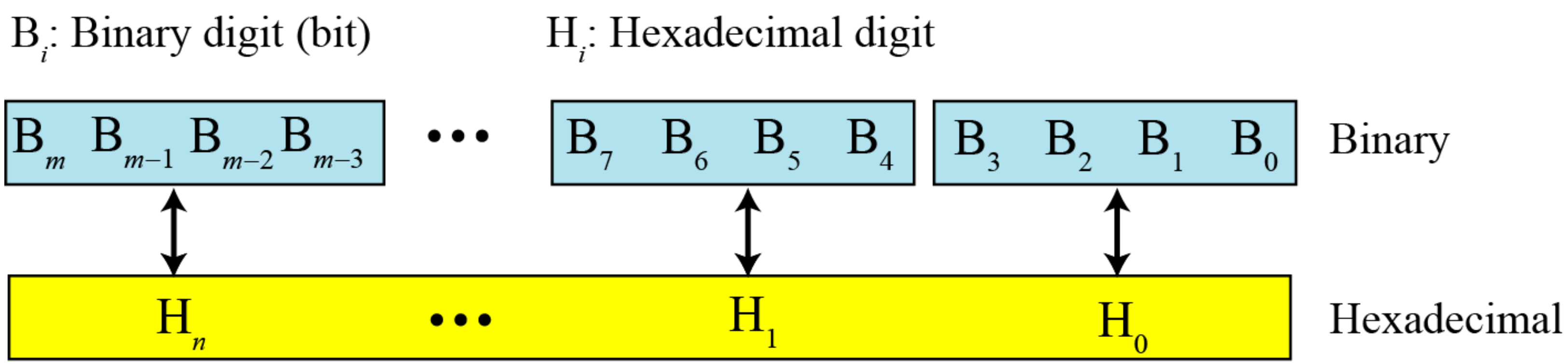


Figure 2.10

Binary to hexadecimal and hexadecimal to binary conversion

Binary-hexadecimal conversion (2)

- Example 2.19 shows the hexadecimal equivalent of the binary number $(110011100010)_2$.

100 1110 0010

  – First, arranging the binary number in 4-bit patterns
  – Using the equivalent of each pattern shown in Table 2.2 on page 25
  – Then, changing the number to hexadecimal: (4E2)16.

Binary-hexadecimal conversion (3)

- Example 2.20: What is the binary equivalent of (24C)16?
- Each hexadecimal digit is converted to 4-bit patterns. The result is $(001001001100)_2$.

$$\mathbf{2 \rightarrow 0010,\ 4 \rightarrow 0100,\ and\ C \rightarrow 1100}$$

Binary-octal conversion (1)

- A relationship between the two bases: three bits in binary is one octal digit

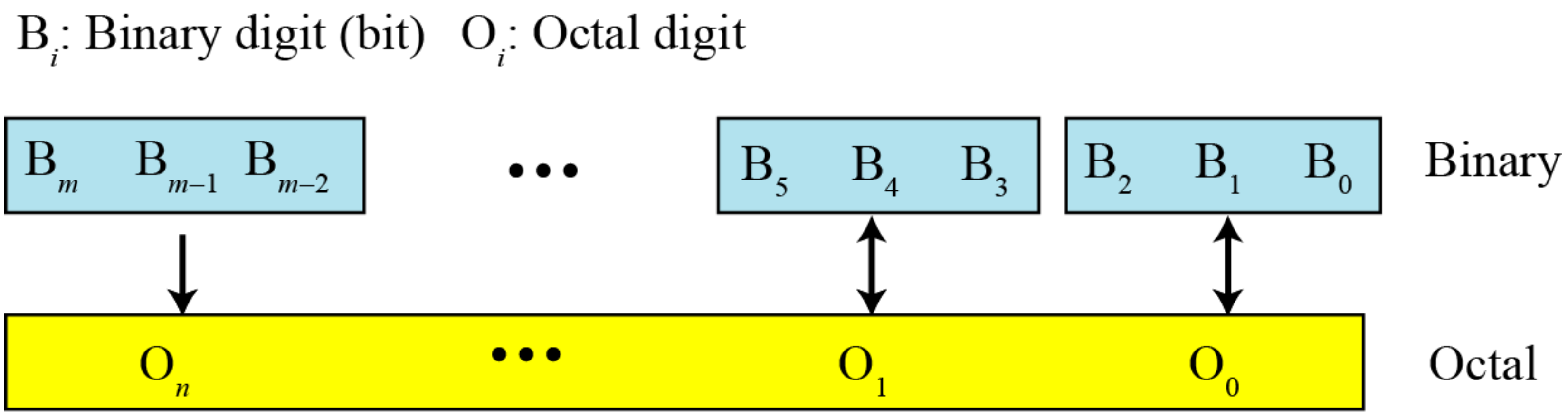


Figure 2.10 Binary to octal and octal to binary conversion

Binary-octal conversion (2)

- Example 2.21 shows the octal equivalent of the binary number $(101110010)_2$.
  - Each group of three bits is translated into one octal digit (Table 2.2). The result is (562)8.

101 110 010

- Example 2.22: What is the binary equivalent of for $(24)_8$?
  - Write each octal digit as its equivalent bit pattern to get. The result is $(010100)_2$.

$$\mathbf{2 \rightarrow 010 \ \ and \ \ 4 \rightarrow 100}$$

Octal-hexadecimal conversion (1)

- Using the binary system as the intermediate system.

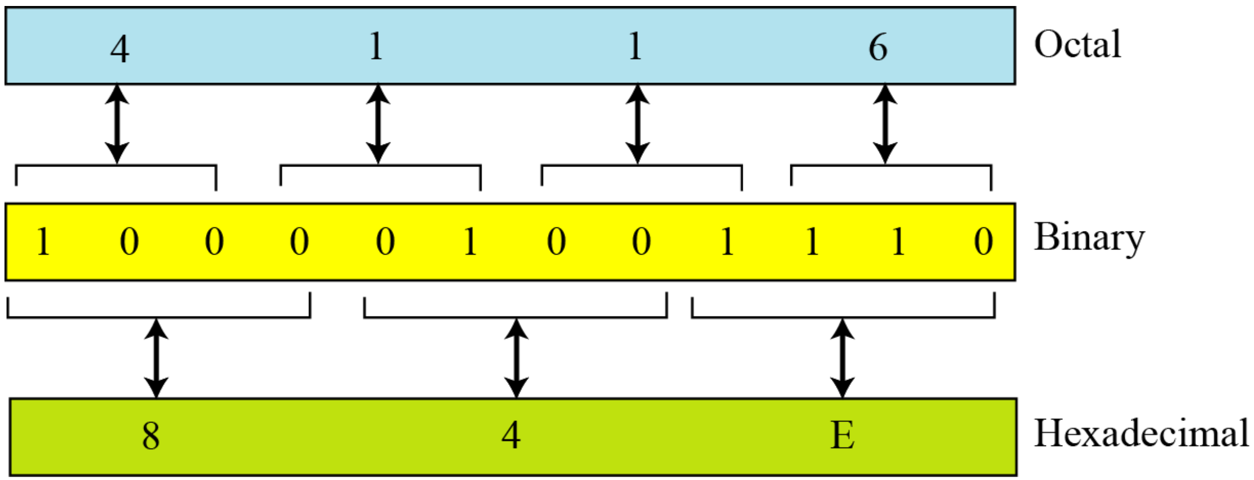


Figure 2.12

Octal to hexadecimal and hexadecimal to octal conversion

Octal-hexadecimal conversion (2)

- Example 2.23 – to find the minimum number of binary digits required to store decimal integers with a maximum of six digits.
  - k = 6, b1 = 10, and b2 = 2. Then

$$x = \lceil k \times (\log b_1 / \log b_2) \rceil = \lceil 6 \times (1 / 0.30103) \rceil = 20.$$

  - The largest six-digit decimal number is 999,999.
  - The largest 20-bit binary number is 1,048,575.
  - The largest 19-bit number is 524287, which is smaller than 999,999.
  - We definitely need twenty bits.

1-3 Nonpositional Number Systems

Overview (1)

- Non-positional number systems are not used in computers.
- A non-positional number system still uses a limited number of symbols in which each symbol has a value.
- We give a short review for comparison with positional number systems.

Overview (2)

- A number is represented as:

$$S_{k-1} \ldots S_2\, S_1\, S_0 \bullet S_{-1}\, S_{-2} \ldots S_{-l}$$

and has the value of:

$$n = \pm \underbrace{S_{k-1} + \ldots + S_1 + S_0}_{\text{Integral part}} + \underbrace{S_{-1} + S_{-2} + \ldots + S_{-l}}_{\text{Fractional part}}$$

Some exceptions to the addition rule, as shown in Example 2.24 (Roman Numerals).

Roman numerals

- Roman numerals is a non-positional number system
  - The set of symbols, S = {I, V, X, L, C, D, M}.
  - Table 2.3 shows the values of each symbol

Table 2.3 Values of symbols in the Roman number system

Symbol	*I*	*V*	*X*	*L*	*C*	*D*	*M*
Value	1	5	10	50	100	500	1000

- To find the value of a number, we need to add the value of symbols subject to specific rules (See Page 34).

Roman numerals and their values

III	→	1 + 1 + 1	=	3
IV	→	5 – 1	=	4
VIII	→	5 + 1 + 1 + 1	=	8
XVIII	→	10 + 5 + 1 + 1 + 1	=	18
XIX	→	10 + (10 – 1)	=	19
LXXII	→	50 + 10 + 10 + 1 + 1	=	72
CI	→	100 + 1	=	101
MMVII	→	1000 + 1000 + 5 + 1 + 1	=	2007
MDC	→	1000 + 500 + 100	=	1600