3 Karnaugh maps and function simplification

3.1 Introduction

One of the objectives of the digital designer when using discrete gates is to keep the number of gates to a minimum when implementing a Boolean function. The smaller the number of gates used, the lower the cost of the circuit. Simplification could be achieved by a purely algebraic process, but this can be tedious, and the designer is not always sure that the simplest solution has been produced at the end of the process.

A much easier method of simplification is to plot the function on a *Karnaugh map* (or 'K-map') and with the help of a number of simple rules to reduce the Boolean function to its minimal form. This particular method is very straightforward up to and including six variables. Above six variables it is better to use a tabulation method such as that due to Quine and McCluskey which, after programming, can be run on a computer.

3.2 Minterms and maxterms

As explained in section 2.5, a *minterm* (sometimes called a 'product term' or 'P-term') of n variables is the logical AND of all n variables where any of the n variables may be represented by the variable itself or its complement. In the case of two variables A and B there are four possible combinations of the variables, and these are tabulated in Figure 3.1. Corresponding to these four combinations of the variables there are four possible minterms which can be obtained as follows. In the first row of the table $A = 0$ and $B = 0$, hence $\bar{A}\bar{B} = 1$. The minterm is formed using the values of the variables which make the value of the minterm equal to 1, hence $m_0 = \bar{A}\bar{B}$. The other three minterms are obtained in the same way.

A	B	Minterms	Maxterms
0	0	$m_0 = \bar{A}\bar{B}$	$M_0 = A + B$
0	1	$m_1 = \bar{A}B$	$M_1 = A + \bar{B}$
1	0	$m_2 = A\bar{B}$	$M_2 = \bar{A} + B$
1	1	$m_3 = AB$	$M_3 = \bar{A} + \bar{B}$

Figure 3.1 *The minterms and maxterms of two variables*

As also explained in section 2.5, a *maxterm* (sometimes called a 'sum term' or 'S-term') of n variables is the logical OR of all n variables where any one of the variables may be represented by its true or complemented form. The maxterms are formed using the values of the variables which make the value of the maxterm equal to 0.

Now, for $A = 0$ and $B = 0$ we have that

$$m_0 = \bar{A}\bar{B} = 1, \text{ and } \bar{m}_0 = \overline{\bar{A}\bar{B}} = 0,$$

giving

$$\bar{m}_0 = M_0 = A + B,$$

i.e. the maxterm is the logical complement of its corresponding minterm. The other three maxterms can be obtained by the same method.

For three variables A, B, and C there are eight possible combinations of the variables and consequently there are eight minterms and eight maxterms. If there are n variables there are 2^n possible combinations of those variables and this leads to 2^n minterms and 2^n maxterms. It is clear that the number of minterms and maxterms rises exponentially with n.

One important property of minterms is that the logical OR of all 2^n minterms is equal to logical 1, i.e.

$$\sum_{i=0}^{2^n-1} m_i = 1$$

The dual of this equation is

$$\prod_{i=0}^{2^n-1} M_i = 0$$

where $\prod$ signifies the Boolean product (AND), so that the logical product of all the maxterms is equal to logical zero. For example, in the case of two variables the logical sum (OR) of all the minterms is given by the expression

$$\begin{aligned} \text{Sum} &= \bar{A}\bar{B} + \bar{A}B + A\bar{B} + AB \\ &= \bar{A}(\bar{B} + B) + A(\bar{B} + B) \\ &= \bar{A} + A \\ &= 1 \end{aligned}$$

Taking the dual of the expression for the sum gives

$$(\bar{A} + \bar{B})(\bar{A} + B)(A + \bar{B})(A + B) = 0$$

and this represents the logical product of all the maxterms of two variables.

3.3 Canonical forms

Also mentioned in section 2.5 is the concept of the *canonical form*, a term used to describe a Boolean function that is written either as a sum of minterms, or as a product of maxterms. For example, using three variables A, B, and C, the equation

$$f(A, B, C) = A(B \oplus C) + \bar{A}\bar{B}C$$

is not written explicitly as a sum of minterms (or a product of maxterms) and so is *not* in canonical form. Simple Boolean algebraic manipulation produces the same function in canonical form written as the logical sum of three minterms:

$$f(A, B, C) = AB\bar{C} + A\bar{B}C + \bar{A}\bar{B}C$$

while the following equation is written as the product of three maxterms and so is also in canonical form:

$$f(A, B, C) = (\bar{A} + \bar{B} + \bar{C})(A + B + \bar{C})(A + \bar{B} + C)$$

3.4 Boolean functions of two variables

There are a specific number of Boolean functions of two variables. Each Boolean function in its canonical form will consist of a certain number of minterms; for example, $f(A, B) = \bar{A}B + A\bar{B}$ is a Boolean function of two variables and contains two of the four available minterms. The total number of Boolean functions of two variables can be obtained in the following manner.

Figure 3.2 shows a table in which the presence of a minterm in a two-variable function is indicated by a 1, and its absence by a 0. For example, if the minterm $\bar{A}\bar{B}$ is included in the expression, its presence will be represented by a 1 in the position of that minterm in the table. If not included, its absence will be indicated by a 0. In the case where all four minterms are absent, this will be indicated by a column of four 0s, as shown in the table, and it follows that the corresponding Boolean function will be $f_0 = 0$.

Minterms	f_0	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}	f_{14}	f_{15}
$m_0 = \bar{A}\bar{B}$	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
$m_1 = \bar{A}B$	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
$m_2 = A\bar{B}$	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
$m_3 = AB$	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

Figure 3.2 *Table for determining all the Boolean functions of two variables*

There are two ways in which the entry in the first row can be allocated: it can be either 0 or 1. There are also two ways in which the entry in the second row can be allocated. When combined with the first row allocation this leads to four ways in which the first two rows can be allocated with 0s and 1s. For four rows, it follows that there are $2^4 = 16$ ways in which the 0s and 1s can be allocated. These allocations are shown in Figure 3.2 and the 16 Boolean functions of two variables can be written down immediately from this table and are tabulated in Figure 3.3.

As the number of variables increases, the number of Boolean functions that can be formed increases rapidly. For three Boolean variables there are $2^8 = 256$ possible Boolean functions, for four variables there are $2^{16} = 65\,536$ possible Boolean functions and for n variables there are $2^{(2^n)}$ possible Boolean functions.

$f_0=0$	False
$f_1=AB$	AND
$f_2=A\bar{B}$	AND (not *B*)
$f_3=A$	Identity
$f_4=\bar{A}B$	AND (not *A*)
$f_5=B$	Identity
$f_6=\bar{A}B+A\bar{B}$	Exclusive OR
$f_7=A+B$	OR
$f_8=\bar{A}\bar{B}=\overline{A+B}$	NOR
$f_9=\bar{A}\bar{B}+AB$	Equality
$f_{10}=\bar{B}$	NOT
$f_{11}=A+\bar{B}$	OR (not *B*)
$f_{12}=\bar{A}$	NOT
$f_{13}=\bar{A}+B$	OR (not *A*)
$f_{14}=\bar{A}+\bar{B}=\overline{AB}$	NAND
$f_{15}=1$	True

Figure 3.3 *The 16 Boolean functions of two variables*