Tutorial letter 201/1/2018

Numerical Methods II

APM3711

Semester 1

Department of Mathematical Sciences

IMPORTANT INFORMATION:

This tutorial letter contains important information about your module.

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ONLY FOR SEMESTER 1 STUDENTS
ASSIGNMENT 01

FIXED CLOSING DATE: 15 MARCH 2018
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Question 1

Use the simple Euler method for the differential equation.

$$\frac{dy}{dx} = \frac{x}{y}, \quad y(0) = 1,$$

with

(a) $h = 0.2$,

(b) $h = 0.1$

to get $y(1)$. Compare your numerical solution with the analytical solution,

$$y^2 = 1 + x^2.$$

SOLUTION

QUESTION 1

For the simple Euler method, the algorithm is

$$y_{n+1} = y_n + hf(x_n, y_n) = y_n + hy'_n$$

where $x_0 = 0,\ y_0 = y(0) = 1$ and

$$f(x, y) = \frac{x}{y}.$$

We will arrange the calculations in the form of a table with columns for $x_n,\ y_n,\ y'_n,\ hy'_n$. The table will be filled one row at the time: We use x_n and y_n to find

$$y'_n = f(x_n, y_n),$$

then multiply by h to get hy'_n, and add this to y_n to get y_{n+1} for the next row.
(a) $h = 0.2$

x_n	y_n	y'_n	hy'_n
0.0	1.0	0.0	0.0
0.2	1.0	0.2	0.04
0.4	1.04	0.384615384	0.076923076
0.6	1.116923077	0.537190082	0.107438016
0.8	1.224361094	0.65340201	0.130680402
1.0	1.355041496		

(b) $h = 0.1$

x_n	y_n	$y'n$	$hy'n$
0.0	1.0	0.0	0.0
0.1	1.0	0.1	0.01
0.2	1.01	0.198019802	0.0198019802
0.3	1.02980198	0.291318142	0.0291318142
0.4	1.058933794	0.37773844	0.037773844
0.5	1.096707638	0.455910018	0.0455910018
0.6	1.14229864	0.525256687	0.0525256687
0.7	1.194824309	0.585860192	0.0585860192
0.8	1.253410328	0.638258662	0.0638258662
0.9	1.317236194	0.683248762	0.0683248762
1.0	1.38556107		

The analytical solution is

$$\begin{aligned} y^2 &= 1 + x^2 \\ &\Rightarrow \quad y = \pm\sqrt{1+x^2} \end{aligned}$$

of which only the "+" sign gives a solution to the initial value problem with $y(0) = 1$. So,

$$y = \sqrt{1+x^2}.$$

At $x = 1$, this gives $y(1) = \sqrt{2} = 1.414213562$. Let us compare our results with this:

Exact solution:	1.414213562	error
Euler with $h = 0.2$:	1.355041496	0.059172066
Euler with $h = 0.1$:	1.38556107	0.028652492

Note that halving the step size from $h = 0.2$ to $h = 0.1$ has halved the error. This is as expected, since the global error in the Euler method is $O(h)$, that is, proportional to h.

Question 2

Solve the differential equation given below by means of the Taylor-series expansion to get the value of y at $x = 1.1$. Use terms up to x^6 and $\Delta x = 0.1$.

$$\begin{aligned} \frac{d^2y}{dx^2} &= xy^2 - 2yy' + x^3 + 4, \\ y(1) &= 1, \quad y'(1) = 2. \end{aligned}$$

SOLUTION

Given

$$\frac{dy^2}{dx^2} = xy^2 - 2yy' + x^3 + 4, \quad y(1) = 1, \quad y'(1) = 2$$

We approximate $y(1.1)$ by the truncated Taylor series

$$y(1.1) \simeq y(1) + (0.1)\, y'(1) + \frac{(0.1)^2}{2} y''(1) + \ldots + \frac{(0.1)^6}{6!} y^{(6)}(1) \qquad (1)$$

The derivatives at $x = 1$ are calculated in the way illustrated on p. 453 [398] of *Gerald*:

$$\begin{aligned} y''(x) &= xy^2 - 2yy' + x^3 + 4, \\ \therefore\ y''(1) &= 1 \cdot 1^2 - 2 \cdot 1 \cdot 2 + 1^3 + 4 = 2 \end{aligned}$$

$$\begin{aligned} y'''(x) &= y^2 + 2xyy' - 2(y')^2 - 2yy'' + 3x^2 + 4, \\ \therefore\ y'''(1) &= 1 + 4 - 8 - 4 + 3 = -4 \end{aligned}$$

$$\begin{aligned} y^{(4)}(x) &= 2yy' + 2yy' + 2x\left((y')^2 + yy''\right) - 4y'y'' - 2y'y'' - 2yy''' + 6x \\ &= 2\left[y(2y' + xy'' - y''') + y'(xy' - 3y'') + 3x\right], \\ \therefore\ y^{(4)}(1) &= 2\left[1(4 + 2 + 4) + 2(2 - 6) + 3\right] = 10 \end{aligned}$$

$$\begin{aligned} y^{(5)}(x) &= 2\left[y'(2y' + xy'' - y''') + y\left(2y'' + y'' + xy''' - y^{(4)}\right)\right. \\ &\quad \left.+\ y''(xy' - 3y'') +\ y'(y' + xy'' - 3y''') + 3\right] \\ &= 2\left[y\left(3y'' + xy''' - y^{(4)}\right) + y'(3y' + 3xy'' - 4y''') - 3(y'')^2 +\ 3\right], \end{aligned}$$

$$\therefore\ y^{(5)}(1) = 2\left[1(6 - 4 - 10) + 2(6 + 6 + 16) - 3(4) +\ 3\right] = 78$$

$$\begin{aligned} y^{(6)}(x) &= 2\left[y'\left(3y'' + xy''' - y^{(4)}\right) + y\left(3y''' + y''' + xy^{(4)} - y^{(5)}\right) + -6y''y'''\right. \\ &\quad \left. y''(3y' + 3xy'' - 4y''') + y'\left(3y'' + 3y'' + 3xy''' - 4y^{(4)}\right)\right] \\ &= 2\left[y\left(4y''' + xy^{(4)} - y^{(5)}\right) + y'\left(12y''4xy''' - 5y^{(4)}\right) + y''(3xy'' - 10y''')\right] \end{aligned}$$

$$\therefore\ y^{(6)}(1) = 2\left[1(-16 + 10 - 78) + 2(24 - 16 - 50) + 2(6 + 40)\right] = -152$$

Substituting these values in (1) we get

$$y(0.1) \simeq -1.209381289.$$

The exact value, rounded to 10 significant numbers, can be calculated by adding more terms of the Taylor series to (1) until it is clear that further addition of terms will not be alter the tenth digit. Note that we have here calculated $y'(0)$ from $y(0)$, $y''(0)$ from $y'(0)$, and so on. Alternatively we could continue further to derive an expression for y', y'' etc. in terms of y and x only, as follows:

$$y'(x) = 3x + (2 + x)\, y,$$

$$y''(x) = 3 + y + (2 + x)\, y' = 3 + y + (2 + x)(3x + (2 + x)\, y)$$

and so on. However, this is unnecessary in this particular problem, and in more complicated differential equations can lead to quite complicated expressions, hence increasing the possibility of errors creeping into the manual calculations.

When implementing the Taylor methods in a computer program, we would of course want to find the expressions in terms of x and y for $y'(x)$, $y''(x)$ and so on, since we would need to evaluate these functions at many different x and y points.

Question 3

Consider the system of coupled second-order differential equations

$$\begin{aligned} u'' - (t+1)(u')^2 + 2uv - u^3 &= \cos t \\ 2v'' + (\sin t)\, u'v' - 6u &= 2t+3 \end{aligned}$$

with initial conditions

$$u(0) = 1, \quad u'(0) = 2, \quad v(0) = 3, \quad v'(0) = 4.$$

Use the second-order Runge-Kutta method with $h = 0.2$ and $a = 2/3$, $b = 1/3$, $\alpha = \beta = 3/2$, to find u, u', v and v' at $t = 0.2$.

SOLUTION

Given

$$\begin{aligned} &u'' - (t+1)(u')^2 + 2uv - u^3 = \cos t \\ &2v'' + (\sin t)\, u'v' - 6u = 2t+3 \end{aligned}$$

$$u(0) = 1, \quad u'(0) = 2, \quad v(0) = 3, \quad v'(0) = 4$$

First, we convert the system of second–order differential equations to a system of first–order differential equations, by introducing two new variables which equal the derivatives of the original variables. Let

$$\begin{aligned} u' &= w, \\ v' &= x, \end{aligned}$$

then the original differential equations can be written as

$$\begin{aligned} w' &= (t+1)\, w^2 - 2uv + u^3 + \cos t \\ x' &= 3u - \frac{1}{2}(\sin t)\, wx + t + \frac{3}{2} \end{aligned}$$

Therefore, the corresponding system of first–order differential equations is

$$\left\{ \begin{array}{llrl} u' = & w & u(0) = & 1 \\ v' = & x & v(0) = & 3 \\ w' = & (t+1)\, w^2 - 2uv + u^3 + \cos t & w(0) = & 2 \\ x' = & 3u - \frac{1}{2}(\sin t)\, wx + t + \frac{3}{2} & x(0) = & 4 \end{array} \right. \tag{1}$$

A system of first–order ordinary differential equations, such as (1), can be solved by any of the methods in Chapter 6 [Chapter 5] of *Gerald* (although the algorithms will become very complicated if the system is large and one of the higher order method is used.)

To see how the algorithms can be adapted to systems such as (1), note first that the system (1) can be written as

$$\begin{cases} u' = & U(t,\ u,\ v,\ w,\ x) & u(0) = & 1 \\ v' = & V(t,\ u,\ v,\ w,\ x) & v(0) = & 3 \\ w' = & W(t,\ u,\ v,\ w,\ x) & w(0) = & 2 \\ x' = & X(t,\ u,\ v,\ w,\ x) & x(0) = & 4 \end{cases} \tag{2}$$

where

$$\begin{cases} U(t,\ u,\ v,\ w,\ x) = & w, \\ V(t,\ u,\ v,\ w,\ x) = & x, \\ W(t,\ u,\ v,\ w,\ x) = & (t+1)\,w^2 - 2uv + u^3 + \cos t, \\ X(t,\ u,\ v,\ w,\ x) = & 3u - \frac{1}{2}(\sin t)\,wx + t + \frac{3}{2}. \end{cases}$$

The basic idea is to adapt the algorithm for solving problems of the form

$$y' = f(t,y)\,, \quad y(t_0) = y_0 \tag{3}$$

to solving problems of the form (2), by replacing y by u, v, w, and x, and replacing f by U, V, W and X. Note that the algorithm is **not** carried out separately for each unknown, but simultaneously for all four.

In this case of the second–order Runge–Kutta method with $a = \dfrac{2}{3}$, $b = \dfrac{1}{3}$, $\alpha = \beta = \dfrac{3}{2}$, the algorithm for (3) is

$$\begin{aligned} y_{n+1} &= y_n + \frac{2}{3}k_1 + \frac{1}{3}k_2, \\ k_1 &= hf(t_n, y_n)\,, \\ k_2 &= hf\left(t_n + \frac{3}{2}h,\ y_n + \frac{3}{2}k_1\right). \end{aligned}$$

Adapted to the system (2), this gives us the algorithm

$$\begin{aligned} u_{n+1} &= u_n + \frac{2}{3}k_{u1} + \frac{1}{3}k_{u2} \\ v_{n+1} &= v_n + \frac{2}{3}k_{v1} + \frac{1}{3}k_{v2} \\ w_{n+1} &= w_n + \frac{2}{3}k_{w1} + \frac{1}{3}k_{w2} \\ x_{n+1} &= x_n + \frac{2}{3}k_{x1} + \frac{1}{3}k_{x2} \end{aligned}$$

$$\begin{aligned}
k_{u1} &= h\,U\left(t_n,\, u_n,\, v_n,\, w_n,\, x_n\right)\\
k_{v1} &= hV\left(t_n,\, u_n,\, v_n,\, w_n,\, x_n\right)\\
k_{w1} &= hW\left(t_n,\, u_n,\, v_n,\, w_n,\, x_n\right)\\
k_{x1} &= hX\left(t_n,\, u_n,\, v_n,\, w_n,\, x_n\right)
\end{aligned}$$

$$\begin{aligned}
k_{u2} &= hU\left(t_n+\frac{3}{2}h,\; u_n+\frac{3}{2}k_{v1},\; v_n+\frac{3}{2}k_{v1},\; w_n+\frac{3}{2}k_{w1},\; x_n+\frac{3}{2}k_{x1}\right)\\
k_{v2} &= hV\left(t_n+\frac{3}{2}h,\; u_n+\frac{3}{2}k_{v1},\; v_n+\frac{3}{2}k_{v1},\; w_n+\frac{3}{2}k_{w1},\; x_n+\frac{3}{2}k_{x1}\right)\\
k_{w2} &= hW\left(t_n+\frac{3}{2}h,\; u_n+\frac{3}{2}k_{v1},\; v_n+\frac{3}{2}k_{v1},\; w_n+\frac{3}{2}k_{w1},\; x_n+\frac{3}{2}k_{x1}\right)\\
k_{x2} &= hX\left(t_n+\frac{3}{2}h,\; u_n+\frac{3}{2}k_{v1},\; v_n+\frac{3}{2}k_{v1},\; w_n+\frac{3}{2}k_{w1},\; x_n+\frac{3}{2}k_{x1}\right).
\end{aligned}$$

Hence, with $h = 0.2$, we calculate $u(0.2)$, $v(0.2)$, $u'(0.2)$ and $v'(0.2)$ from the given initial values as follows:

$$\begin{aligned}
k_{u1} &= (0.2)\;U\left(0,\,1,\,3,\,2,4\right) = (0.2)\,(2) = 0.4\\
k_{v1} &= (0.2)\;V\left(0,\,1,\,3,\,2,4\right) = (0.2)\,(4) = 0.8\\
k_{w1} &= (0.2)\;W\left(0,\,1,\,3,\,2,4\right)\\
&= (0.2)\left[(0+1)\,2^2 - 2\cdot 1\cdot 3 + 1^3 + 1\right] = 0\\
k_{x1} &= (0.2)\;X\left(0,\,1,\,3,\,2,4\right)\\
&= (0.2)\left(3\cdot 1 - (0.5)\cdot 0\cdot 2\cdot 4 + 0 + 1.5\right) = 0.9
\end{aligned}$$

$$
\begin{aligned}
k_{u2} &= (0.2)\ U\left(0+\frac{3}{2}(0.2),\ 1+\frac{3}{2}(0.4),\ 3+\frac{3}{2}(0.8),\ 2+\frac{3}{2}(0),\ 4+\frac{3}{2}(0.9)\right) \\
&= (0.2)\ U(0.3,\ 1.6,\ 4.2,\ 2.0,\ 5.35) \\
&= (0.2)\cdot 2 = 0.4
\end{aligned}
$$

$$
\begin{aligned}
k_{v2} &= (0.2)\ V(0.3,\ 1.6,\ 4.2,\ 2.0,\ 5.35) \\
&= (0.2)\cdot 5.35 = 1.07
\end{aligned}
$$

$$
\begin{aligned}
k_{w2} &= (0.2)\ W(0.3,\ 1.6,\ 4.2,\ 2.0,\ 5.35) \\
&= (0.2)\left[(0.3+1)\,2^2 - 2(1.6)(4.2) + (1.6)^3 + \cos(0.3)\right] \\
&= -0.6377
\end{aligned}
$$

$$
\begin{aligned}
k_{x2} &= (0.2)\ X(0.3,\ 1.6,\ 4.2,\ 2.0,\ 5.35) \\
&= (0.2)\left[3(1.6) - (0.5)\cdot\sin(0.3)\cdot 2\cdot(5.35) + 0.3 + 1.5\right] \\
&= 1.0038
\end{aligned}
$$

$$
\begin{aligned}
u(0.2) &= 1+\frac{2}{3}(0.4)+\frac{1}{3}(0.4) \approx 1.4 \\
v(0.2) &= 3+\frac{2}{3}(0.8)+\frac{1}{3}(1.07) \approx 3.89 \\
u'(0.2) &= w(0.2) = 2+\frac{2}{3}(0)+\frac{1}{3}(-0.6377) \approx 1.79 \\
v'(0.2) &= x(0.2) = 4+\frac{2}{3}(0.9)+\frac{1}{3}(1.0038)\,9 \approx 4.93
\end{aligned}
$$

Question 4

Consider the boundary value problem

$$\frac{d^2y}{dx^2} + 2xy = 2, \quad 0 \le x \le 2.$$

Set up the set of equations to solve this problem by the method of finite differences when $h = \Delta x = \frac{1}{2}$ is used, in each of the following cases of boundary conditions. (You do not have to solve the equations.)

(a) $y(0) = 5,$

$y(2) = 10.$

(b) $y'(0) = 2,$

$y(2) = 0.$

SOLUTION

To solve the boundary value problem

$$\frac{d^2y}{dx^2} + 2xy = 2, \quad 0 \le x \le 2,$$

by the method of finite differences we convert the differential equation into a difference equation. This is achieved by replacing derivatives by appropriate finite-difference formulas at chosen grid points. The following rules should be taken into account:

- Central-difference formulas should be used, since they are more accurate than forward or backward approximations.
- We are using the difference equation to approximate the differential equation **at some grid point**. Remember that each difference formula approximates the value of a derivative at a certain grid point. Accordingly, all the terms of the differential equation must be replaced by the corresponding difference quantities consistently, such that they all approximate the original quantities at the same given grid point.

Let x_i denote the grid points, $h = x_{i+1} - x_i$, and let y_i denote the approximate value of $y(x_i)$. We will use the following approximations:

$$\left.\frac{\partial^2 y}{\partial x^2}\right|_{x=x_i} \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

$$\left. y\right|_{x=x_i} \approx y_i$$

(and of course $\left. x\right|_{x=x_i} = x_i$).
Thus the differential equation can be approximated at a grid point x_i by the difference equation

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + 2x_i y_i = 2.$$

When $h = \frac{1}{2}$, we get

$$2y_{i-1} + (x_i - 4)\, y_i + 2y_{i+1} = 1. \qquad (*)$$

The grid in this problem consists of following the points:

$$x_0 = 0, \;\; x_1 = 0.5, \;\; x_2 = 1.0, \;\; x_3 = 1.5, \;\; x_4 = 2.0.$$

Equation $(*)$ needs to be applied at all grid points at which the value of the function y is unknown. This usually includes all grid points in the interior of the region, but whether the values of y at the boundaries of the region are known or not depends on the boundary conditions.

(a) Boundary conditions $y(0) = 5$, $y(2) = 10$:
In this case, $y_0 = 5$ and $y_4 = 10$ are both know, and therefore we only need to apply (*) at the interior grid points x_1, x_2 and x_3. We get the three equations

$$\begin{cases} 2y_0 + (x_1 - 4)\, y_1 + 2y_2 = 1 \\ 2y_1 + (x_2 - 4)\, y_2 + 2y_3 = 1 \\ 2y_2 + (x_3 - 4)\, y_3 + 2y_4 = 1 \end{cases}$$

which, after substituting the values of x_1, x_2, x_3, y_0 and y_4 give

$$\begin{cases} -3.5y_1 + 2y_2 = -9 \\ 2y_1 - 3y_2 + 2y_3 = 1 \\ 2y_2 - 2.5y_3 = -19 \end{cases}$$

(b) Boundary conditions $y'(0) = 2$, $y(2) = 0$:
Now, $y_4 = 0$ but y_0 is not known, and therefore we need to apply (*) at x_0 as well as at the interior grid points x_1, x_2 and x_3. Since the equation $(*)$ refers to the value of y_{i-1} at a grid point $x_{i-1}$to the left of the "current" grid point x_i, to be able to apply $(*)$ at the grid point x_0 we need to also include the value for y at $x_{-1} = -0.5$, which is outside the domain of the differential equation. To handle this problem, we will introduce a fictitious point, y_{-1} to represent the value $y(-0.5)$. We will write out the set of equations using this extra point, and then use the derivative boundary value to eliminate it.

The application of $(*)$ at x_0, x_1, x_2 and x_3 gives the set of equations

$$\begin{cases} 2y_{-1} + (0 - 4)\, y_0 + 2y_1 &= 1 \\ 2y_0 + (0.5 - 4)\, y_1 + 2y_2 &= 1 \\ 2y_1 + (1.0 - 4)\, y_2 + 2y_3 &= 1 \\ 2y_2 + (1.5 - 4)\, y_3 + 2y_4 &= 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} 2y_{-1} - 4.0\, y_0 + 2y_1 = 1 \\ 2y_0 - 3.5\, y_1 + 2y_2 = 1 \\ 2y_1 - 3.0\, y_2 + 2y_3 = 1 \\ 2y_2 - 2.5\, y_3 = 1 \end{cases} \qquad (**)$$

Next, we apply the boundary condition

$$y'(0) = 2. \qquad (1)$$

Using central differences to approximate the derivative $y'(0)$ at $x_0 = 0$, (1) can be expressed as the difference relation

$$\frac{y_1 - y_{-1}}{2h} = 2$$

$$\Leftrightarrow \quad y_{-1} = y_1 - 2.$$

We substitute this into $(**)$, and get the set of equations

$$\begin{cases} -4\, y_0 + 4y_1 = 5 \\ 2y_0 - 3.5\, y_1 + 2y_2 = 1 \\ 2y_1 - 3\, y_2 + 2y_3 = 1 \\ 2y_2 - 2.5\, y_3 = 1 \end{cases}$$

Question 5

The predictor and corrector formulas of the Adam-Moulton method are:

$$y_{n+1} = y_n + \tfrac{h}{24}\left(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}\right) + \tfrac{251}{720}h^5 y^5\left(\zeta_1\right)$$
$$y_{n+1} = y_n + \tfrac{h}{24}\left(9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}\right) - \tfrac{19}{720}h^5 y^{(5)}\left(\zeta_2\right).$$

Apply the Adams-Moulton method to calculate the approximate value of $y\,(0.8)$ and $y\,(1.0)$ from the differential equation

$$y' = t + y$$

and the starting values

t	**y(t)**
0.0	0.95
0.2	0.68
0.4	0.55
0.6	0.30

Use 3 decimal digits with rounding at each step.

SOLUTION

$y' = t + y = f\,(t.y)$: The given starting values are

$$\begin{array}{lllll} n = 1 & t_1 = 0.0 & y_1 = 0.95 & f_1 = 0.95 \\ n = 2 & t_2 = 0.2 & y_2 = 0.68 & f_2 = 0.88 \\ n = 3 & t_3 = 0.4 & y_3 = 0.55 & f_3 = 0.95 \\ n = 4 & t_4 = 0.6 & y_4 = 0.30 & f_4 = 0.9 \end{array}$$

At $t = 0.8$, the predictor is

$$\begin{aligned} y_5 &= y_4 + \frac{0.2}{24}\left(55f_4 - 59f_3 + 37f_2 - 9f_1\right) \\ &= 0.3 + \frac{0.2}{24}\left(55\,(0.9) - 59\,(0.95) + 37\,(0.88) - 9\,(0.95)\right) \\ &= 0.4455, \quad f_5 = 1.2455 \end{aligned}$$

and the corrector is

$$\begin{aligned} y_5 &= y_4 + \frac{h}{24}(9f_5 + 19f_4 - 5f_3 + f_2) \\ &= y_4 + \frac{0.2}{24}\left(9\,(1.2455) + 19\,(0.9) - 5\,(0.95) + 0.88\right) \\ &= 0.5037, \qquad f_5 = 1.3037. \end{aligned}$$

1. (a) At $t = 1.0$, the predictor is:

$$
\begin{aligned}
y_6 &= y_5 + \frac{0.2}{24}\left(55f_5 - 59f_4 + 37f_3 - 9f_2\right) \\
&= 0.5037 + \frac{0.2}{24}\left(55\,(1.3037) - 59\,(0.9) + 37\,(0.95) - 9\,(0.881)\right) \\
&= 0.8856, \qquad f_6 = 1.8856
\end{aligned}
$$

and the corrector is:

$$
\begin{aligned}
y_6 &= y_5 + \frac{h}{24}(9f_6 + 19f_5 - 5f_4 + f_3) \\
y_6 &= 0.5037 + \frac{0.2}{24}\left(9\,(1.8856) + 19\,(1.3037) - 5\,(0.9) + 0.95\right) \\
&= 0.822.
\end{aligned}
$$

Question 6

(a) Given the truncated power series

$$p(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \frac{x^5}{5}$$

and the Chebyshev polynomial

$$T_5(x) = 16x^5 - 20x^3 + 5x.$$

(i) What is the purpose of economizing a power series?

(ii) Economize the power series $p(x)$.

(iii) What is the maximum value of $|T_5(x)|$ in the interval [-1; 1]?

(iv) If $q(x)$ denotes the economized series obtained in (ii), what is the maximum value of $|p(x) - q(x)|$ in the interval [-1; 1]?

(b) We want to find a Padé approximation $R_6(x)$, with denominator of degree 3, to the function

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + ...$$

(i) Why is it often preferred to approximate a function by a rational function rather than by a polynomial?

(ii) Write down $R_6(x)$ with coefficients a_i and b_i.

(iii) Set up the equations to find the coefficients, but do NOT solve the system.

SOLUTION

(a)(i) An economized power series gives almost the same accuracy as the original one, with fewer terms.

(ii) $p(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \frac{x^5}{5}$

The economized power series is

$$\begin{aligned} & p(x) - \frac{1}{5 \cdot 16} T_5(x) \\ = \;& 1 - x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \frac{x^5}{5} + \frac{1}{5 \cdot 16}\left(16x^5 - 20x^3 + 5x\right) \\ = \;& 1 - \frac{15}{16}x + \frac{x^2}{2} - \frac{7x^3}{12} + \frac{x^4}{4} \end{aligned}$$

(iii) $\max\limits_{[-1,1]} |T_5(x)| = 1$

(iv) $q(x) = p(x) - \frac{1}{5 \cdot 16} T_5(x) = p(x) - \frac{1}{80} T_5(x)$

$\therefore \quad |p(x) - q(x)| = \frac{1}{80}|T_5(x)| \leq \frac{1}{80} = 0.0125$

when $\quad -1 \leq x \leq 1.$

(b)(i) A rational approximation can give greater accuracy for the same number of coefficients (or, the same accuracy with fewer coefficients).

(ii) R_6 with denominator of degree 3 is

$$R_6(x) = \frac{a_0 + a_1x + a_2x^2 + a_3x^3}{1 + b_1x + b_2x^2 + b_3x^3}$$

(iii) $e^{x^2} - R_6(x)$

$$= \frac{(1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \ldots)(1 + b_1x + b_2x^2 + b_3x^3) - (a_0 + a_1x + a_2x^2 + a_3x^3)}{(1 + \ldots + b_3x^3)}$$

In the numerator, set the coefficients of terms $1, x, \ldots$ up to x^6 equal to zero:

$$\begin{aligned} 1 :\;& 1 - a_0 = 0 \\ x :\;& b_1 - a_1 = 0 \\ x^2 :\;& b_2 + 1 - a_2 = 0 \\ x^3 :\;& b_3 + b_1 - a_3 = 0 \\ x^4 :\;& b_2 + \tfrac{1}{2} = 0 \\ x^5 :\;& b_3 + \tfrac{1}{2}b_1 = 0 \\ x^6 :\;& \tfrac{1}{2}b_2 + \tfrac{1}{6} = 0. \end{aligned}$$