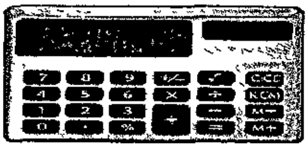
UNIVERSITY EXAMINATIONS



UNIVERSITEITSEKSAMENS

UNISA university of south africa

APM1514

May/June 2013

MATHEMATICAL MODELLING

Duration 2 Hours

100 Marks

EXAMINERS :
FIRST DR EM RAPOO
SECOND MR AS KUBEKA

Use of a non-programmable pocket calculator is permissible.

Closed book examination

This examination question paper remains the property of the University of South Africa and may not be removed from the examination venue

This paper consists of 3 pages

You should answer **ALL** the questions

1 Assume that a proportionally growing population has a birth rate of 0 1 births per population member per year and a death rate of 0 2 deaths per population member per year At the beginning of each year, 20 000 new members enter the population

(a) Write down the difference equation to model the population Use P_n to denote the number of population members present at the end of year n

(b) If the current population size at year n is 0, is the population increasing or decreasing? Justify your answer!

(c) If the current population size at year n is 100 000, is the population increasing or decreasing? Justify your answer!

(d) If the current population size at year n is 500 000, is the population increasing or decreasing? Justify your answer!

(e) Is there any initial population size $P_0 \geq 0$ from which the population will die out? [15%]

2 (a) Find the solution to the following

$$\frac{dx}{dt} = 1 - 2x, \quad x(2) = \frac{1}{2}$$

(b) Which of the following statements are true and which are false? Justify your answers!

(i) If a radioactive substance decays from M to N in time T then it decays from M to $\frac{1}{2}N$ in time $2T$

(ii) In the Malthusian model, if $k > 0$ then the equilibrium point is positive

(iii) In a predator-prey system, neither the predator nor the prey can ever die out

[TURN OVER]

(iv) For an autonomous differential equation

$$\frac{dx}{dt} = F(x),$$

equilibrium points are the points where $F(x)$ neither increases nor decreases.

(v) Assume that populations A and B both grow according to the Malthusian model both with non-zero initial populations If $k_A > k_B$ then regardless of the initial population sizes, eventually population A will be larger than population B

(vi) Assume that populations A and B both grow according to the Malthusian model both with non-zero initial populations If $k_A > k_B$ then at any given time t, population A grows faster than population B [20%]

3 (a) The population of a town obeys the Malthusian model, with $k = -0\ 023$ and time measured in years

(i) How long does it take for the population to reach 50% of its original size?

(ii) If the town has 10 people today, when was there 10 000 people?

(iii) How long does it take for the population to decrease by 1000 people, if the initial population was 20 000?

(b) A logistically growing population has growth constants $a = 0.01$, $b = 0\ 005$, with t measured in years, and the initial population size is $P_0 = 20$ The solution in the logistic model is given by

$$P(t) = \frac{a}{b + \left(\frac{a}{P_0} - b\right) e^{-at}}$$

(i) What is the initial rate of growth of the population?

(ii) For which population size does the population grow at the fastest possible rate? [17%]

4 The spread of an infection in a community is modelled by the following differential equation.

$$\frac{dX}{dt} = aX(X - 200),$$

where $X(t)$ is the number of infections after t days and a is a positive constant

(a) Draw the phase line of the system

(b) Use the phase line to answer the following questions Justify your answers!

(i) In this model, will the number of infected people always grow without bound?

(ii) In this model, if there are initially only few infected people, will the infection then die out (i e $X(t)$ will approaches zero)?

(iii) Is this a special case of the logistic model? [14%]

[TURN OVER]

5 A cake is baked in an oven, the temperature of which is kept at 220°. Initially the cake has the temperature 25° Assume that Newton's law of cooling/heating up applies

(a) Explain why we do not have enough information to find the value of the constant k

(b) If $k = 0\ 08$ (time measured in minutes), how long does it take for the cake to reach the temperature of 100°?

(c) If $k = 0\ 08$ (time measured in minutes), what is the rate of change of the temperature of the cake after 60 minutes?

(d) At what time is the rate of change of the temperature at its highest? [14%]

6. Draw the phase diagram of the following system

$$\frac{dx}{dt} = 3 - x$$

$$\frac{dy}{dt} = 3xy$$

You should include all four quadrants of the xy–plane For full marks, all the following must be given in your answer The isoclines and all the equilibrium points, and the signs of dx/dt and dy/dt in different parts of the xy–plane In addition, the following must be correctly and clearly annotated in your phase diagram. The coordinate axes, all the isoclines, all the equilibrium points, the allowed directions of motion (both vertical and horizontal) in all the regions into which the isoclines divide the xy–plane, direction of motion along isoclines, where applicable, examples of allowed trajectories in all regions, and examples of trajectories crossing from a region to another, whenever such a crossing is possible

Specify also which equilibrium points of the system are stable and which are unstable Justify your answers!

You do NOT have to list the possible outcomes of the system, beyond the stability of the equilibrium points [20%]