Tutorial Letter 101/3/2018

Linear Mathematical Programming

DSC2605

Semester 1 & 2

Department of Decision Sciences

Important information:

Please register on myUnisa, activate your myLife e–mail addresses and make sure that you have regular access to myUnisa module website, DSC2605–2018–S1/S2, as well as your group website.

Note: This is an online module, and therefore your module is available on myUnisa. This is the only document you will receive in printed format.

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Contents

1 Introduction and Welcome

Dear student

It is a pleasure to welcome you to the module **Linear Mathematical Programming** (DSC2605) offered by the Department of Decision Sciences. We hope that you will find this module interesting and that you will complete it successfully. It is essential that you read this Tutorial Letter 101 very carefully as it contains important information about the module, the assignments and their due dates.

1.1 Tutorial matter

The Despatch Department should supply you with the study guide and Tutorial Letter 101 (this tutorial letter) for this module. Tutorial Letter 101 can also be found on *my*Unisa under "**Official Study Material**". The study guide, Tutorial Letter 201 containing the solutions to the compulsory assignments, and any other tutorial letters that may be necessary will be found under "**Additional Resources**".

You will also receive important announcements regarding the module and study material through your *my*Life email address. You are therefore required to access your *my*Life email address regularly to ensure that you do not miss any information.

1.2 Computer requirements

Access to a computer is compulsory. In this module, we require you to use LINGO, LINDO and MAXIMA software packages. They are all WINDOWS driven. LINDO and LINGO are available on the internet at `http://www.lindo.com` and MAXIMA at `http://maxima.sourceforge.net/download.html`.

The software LINDO and LINGO are also included in the prescribed book. We suggest that you install the software packages as soon as possible and start familiarizing yourself with them immediately.

Guidelines on the use of LINDO and LINGO are available in the prescribed book, the study guide and on *my*Unisa under "**Additional Resources**". A **Maxima** tutorial is available on *my*Unisa under "**Additional Resources**". **Maxima** is a handy tool to check or perform your calculations, especially for matrix calculations. You may install it online.

2 Purpose and Outcomes

This section contains the purpose of as well as the learning outcomes and the assessment criteria for the module. The assessment criteria specify how you will be assessed to determine whether you have attained the learning outcomes.

2.1 Purpose

Those who complete this module will be able to model and solve optimization problems in the economic and financial environment with techniques of linear programming and linear algebra. Qualifying students will be able to:

– Do matrix operations.

– Solve systems of linear equations.

– Formulate a practical problem as a linear programming model.

– Solve linear programming (LP) models by using an appropriate method and interpret their solutions.

– Recognize different types of solutions of the LP models.

– Use computer software packages.

2.2 Outcomes

Learning outcome 1: "*Learners are able to do basic matrix operations.*"

Assessment criteria

1. Understand what matrices and vectors are.
2. Do matrix operations.
3. Write a system of linear equations as a matrix.
4. Use the Gauss-Jordan method to solve systems of linear equations.
5. Explain linear independence and linear dependence.
6. Find the inverse of a matrix.
7. Find the reduced row echelon form of a matrix
8. Calculate the determinant of a matrix.
9. Find the transpose of a matrix.
10. Use a computer package to perform matrix calculations.

Learning outcome 2: "*Learners are able to formulate linear programming models*".

Assessment criteria

1. Give the properties and assumptions of linear programming.
2. Formulate a linear programming model of a given practical problem.

Learning outcome 3: "*Learners are able to solve linear programming problems.*"

Assessment criteria

1. Use the graphical method to solve linear programming problems in two variables.
2. Use the simplex or Big M methods to solve linear programming problems.
3. Recognize infeasible, multiple, unbounded or degenerate solutions, and redundant or binding constraints from the graphical, computer or simplex solution of a linear programming problem.
4. Use a computer package to solve any given linear programming problem.

3 Contact details

3.1 Lecturer contact details

You may contact the lecturer during office hours for any academic queries about the module. The lecturer's contact details are given below:

– Lecturer: Mr. Marc M Mpanda

– Email: mpandmm@unisa.ac.za

– Tel: +27 12 433 4716.

– Office: 4-34, Club One, Hazelwood, Pretoria East.

Any changes to the contact details will be communicated on *my*Unisa or via an SMS.

You may contact the lecturer by telephone or by email (**preferable**). Please include your student number and the module code when you send an email. You can also see the lecturer personally but make an appointment beforehand.

3.2 Department contact details

The Department of Decision Sciences can be contacted by

– Email: qm@unisa.ac.za

– Tel : +27 12 433 4684

4 Module Related Resources

4.1 Prescribed book

The following book has been prescribed for this module and you must buy it:

- **Title:** OPERATIONS RESEARCH: Applications and Algorithms.
- **Author:** Winston, Wayne L.
- **Edition, Year:** Fourth edition, 2004.
- **Publisher:** Thomson.
- **ISBM:** 9780534423629

We will refer to this book as *Winston* in the remainder of this tutorial letter. *Winston* contains a CD with several computer software packages. You will need the packages LINDO or LINGO for this module.

Prescribed books can be obtained from the University's official booksellers. If you have difficulty in locating your book(s) at these booksellers, please contact the Prescribed Book Section.

4.2 Recommended book

You may also use the following book for this module.

- **Title:** Introduction to Operation Research.
- **Author:** Hillier, FS
- **Edition, Year:** tenth edition, 2015.
- **Publisher:** McGraw-Hill Education
- **ISBM:** 9781259253188

4.3 Study approach

We suggest that you approach the study material as follows:

- Learn to use the **Maxima**, LINDO and LINGO software packages **before** you start working through the study material.
- Work through a learning unit in the study guide and study the corresponding sections in *Winston*. Each learning unit contains examples and exercises. Solutions of some exercises will be posted later on *my*Unisa.

– Do the evaluation exercises that have been set on the learning unit and evaluate your answers using the model solutions.

– Answer the compulsory Assignments 01, 02 and 03, and submit them to reach Unisa before the due dates.

– Prepare for the examination.

All this work must be done in one semester, the duration of which is about four months. This is a very limited period of time and you will have to plan your studies carefully. Experience has shown that many students underestimate this module and do not set aside sufficient time for it. Do not delay, **start working immediately**!

4.4 Syllabus

The syllabus for this module consists of two parts:

- Part 1 – Linear algebra
- Part 2 – Linear programming (LP).

Further information about the syllabus, guidelines for studying the module and the study material that you must study to cover the syllabus (with sections from the prescribed book, *Winston*) can be found on *my*Unisa under “ **Additional resources**”.

5 Assessment

The assessment consists of formative assessments (assignments) and summative assessment (written exam).

5.1 Assessment Plan

This tutorial letter contains assignments for both semesters; three assignments for the first semester and another three for the second semester.

Important Notes:

– Assignment 1 contributes 20% towards your semester mark,

– Assignment 2 contributes 60% towards you semester mark and

– Assignment 3 contributes 20% towards your semester mark.

Therefore, your semester mark is obtained by using the following formula:

$$\boldsymbol{Semester\ Mark} = 0,2 \times ASS1 + 0,6 \times ASS2 + 0,2 \times ASS3.$$

Your semester mark contributes 20% towards your final mark. Your final mark will be

$$\textbf{\textit{Final Mark}} = 0,2 \times \textbf{\textit{Semester Mark}} + 0,8 \times \textbf{\textit{Exam Mark}}.$$

Suppose you obtain 46% for the exam and 70% for your semester mark. Your final mark is then calculated as follows:

$$\begin{aligned} \textbf{\textit{Final Mark}} &= (0,2 \times 70 \ + \ 0,8 \times 46)\% \\ &= (14 + 36,8)\% \\ &= 50,8\% \\ &= 51\% \ \text{(rounded to the nearest percent).} \end{aligned}$$

You can thus see how important it is to obtain a good semester Mark !

The unique assignment numbers and the due dates of each assignment for both semesters are summarized in the following table:

		Unique Numbers	Due Dates
SEMESTER 1	Assignment 1	719181	9 March 2018
	Assignment 2	840490	13 April 2018
	Assignment 3	746438	26 April 2018
SEMESTER 2	Assignment 1	815996	3 August 2018
	Assignment 2	821150	7 September 2018
	Assignment 3	889072	21 September 2018

You must submit at least one assignment for this module to obtain admission to the examination. Multiple choice assignments should be submitted electronically via *my*Unisa. Written assignments should also be submitted on *my*Unisa and **must be in pdf format**.

5.2 Examination

The examination will be a two – hour closed – book assessment. Programmable pocket calculator will be allowed. You **must** write the examination in **May/June** if you are registered for the **first** semester, and in **October/November** if you are registered for the **second** semester.

Please make sure that you know the correct **date, time** and **venue** of your examination, and plan your studies accordingly. **We will not accept any mistakes regarding the date, time, or venue, as an excuse for a supplementary examination at the end of the next semester.**

6 Assignments

6.1 First Semester Assignments

ASSIGNMENT 1
Only for Semester 1 students
Due date: 9 March 2018
Unique Assignment Number: 719181

Important Notes:

◇ ***Assignment 1 is compulsory and you must submit it to reach UNISA before the due date.*** *No late assignments will be marked.*

◇ ***Assignment 1 consists of 4 questions and covers learning units 1 to 4 of the study guide.***

◇ *This assignment contributes **20%** towards your semester mark.*

Question 1: Discussion on *my*Unisa **[7]**

Marks will be awarded for your participation and contribution on the Discussion Forum on *my*Unisa by the due date. You will thus need to visit the module site on *my*Unisa at least one hour per week.

(a) Write down topics that you have created under a specific discussion forum. (2)

(b) State clearly all contributions in different topics created either by you, your fellow students or your lecturers. (5)

Question 2 **[42]**

Consider the following matrices:

$$A = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & 2 \\ -1 & 3 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & p & p^2 \\ 1 & q & q^2 \\ 1 & r & r^2 \end{bmatrix},$$

where p, q and r are real parameters.

(a) Find the inverse of the matrix A by using the Gauss–Jordan method. (10)

(b) Prove that the determinant $|B| = (r - q)(r - p)(q - p)$. (6)

Hint: The matrix B is well-known as the "*Vandermonde matrix*". Use different properties of determinants to find $|B|$.

(c) Assume that X is an unknown square matrix of order 3. For all $p \neq q \neq r$, find all entries of the matrix X such that $2A - BX = 2I_3$. Here I_3 represents the identity matrix of order 3. (20)

Hint: Solve the equation in matrix form and find all entries of the matrix X. The inverse B^{-1} may be computed by using the cofactor method with the determinant $|B|$ given in (b).

(d) Find an equivalent matrix B' obtained by performing the following series of elementary row operations on the matrix B. (6)

$$\begin{aligned} R_2' &= \frac{1}{q}R_2 \\ R_1' &= R_1 - pR_2' \\ R_3' &= R_3 - rR_2'. \end{aligned}$$

Question 3 **[16]**

Consider the following system of linear equations.

$$\begin{array}{ccccccccc} x & + & y & + & z & + & w & = & 6 \\ x & & & + & z & + & w & = & 4 \\ x & & & + & z & & & = & 2 \end{array}$$

(a) List all leading and free variables while solving the linear system above by using the Gauss-Jordan elimination method.

(b) Find the general solution of the system and write down the solution set clearly.

Question 4 **[20]**

Consider the following system of linear equations.

$$\begin{array}{rcrcrcl} x & + & y & - & z & = & a \\ x & - & y & + & 2z & = & b \\ 3x & + & y & & & = & c \end{array}$$

(a) Find the general solution when the system above is homogeneous and deduce a nontrivial solution.

(b) Find all values of a, b and c for which the linear system above has :

(b_1) no solution.

(b_2) a unique solution.

(b_3) infinitely many solutions.

Total: 85 Marks

ASSIGNMENT 2
Only for Semester 1 students
Due date: 13 April 2018
Unique Assignment Number: 840490

Important Notes:

- ♢ ***Assignment 2 is compulsory and you must submit it to reach UNISA before the due date.*** *No late assignments will be marked.*
- ♢ ***Assignment 2 consists of 5 questions and covers learning units 5 to 9 of the study guide.***
- ♢ *This assignment contributes **60%** towards your semester mark.*

Question 1 [7]

Marks will be awarded for your participation and contribution on the Discussion Forum on *my*Unisa by the due date. You will thus need to visit the module site on *my*Unisa at least one hour per week.

(a) Write down topics that you have created under a specific discussion forum. (2)

(b) State clearly all contributions in different topics created either by you, your fellow students or your lecturers. (5)

Question 2 [13]

A railway company wishes to purchase a new fleet of railway carriages. The fleet must have the capacity to transport at least 3 400 tons of coal and at least 1800 tons of lumber at any given moment. Furthermore, at least a quarter of the carriages must be able to transport coal. The company can buy three types of carriages, models A, B and C.

- One carriage of model A costs 6 million rand and can transport 25 tons of coal.
- One carriage of model B costs 4 million rand and can transport 8 tons of lumber.
- One carriage of model C costs 8 million rand and can transport 18 tons of coal or 10 tons of lumber (not both simultaneously).

(a) Define all decision variables clearly. (2)

(b) Formulate this problem as a linear programming model that gives the least cost for which the company can buy a fleet with the desired capacity. (6)

(c) Write LINGO's codes that will solve the problem formulated in (b). Run these codes and attach the Solution Report. (3)

(d) Deduce the optimal solution (if it exists). (2)

Question 3 **[19]**

An oil company makes two types of blend (Blend 1 and Blend 2) by mixing three types of oil (Oil 1, Oil 2 and Oil 3). The cost per liter of each oil and the daily availability are given in Table 1 below:

Table 1: Costs and daily availability

Oil	**Cost (R/Liter)**	**Quantity available (Liters)**
1	6	8 000
2	8	12 000
3	10	16 000

The following requirements should be taken into consideration.

1. Blend 1 must contain at most 40% of Oil 1, at least 30% of Oil 2 and at most 30% of Oil 3.
2. Blend 2 must contain at least 40% of Oil 1, at most 50% of Oil 2 and at least 30% of Oil 3.
3. Each liter of Blend 1 can be sold for R40 and each liter of Blend 2 can be sold for R50. In addition, at leat 12 000 liters of each blend must be produced.

Let $X_{ij} (i = 1, 2, 3$ and $j = 1, 2)$ be the decision variables representing the amount of Oil i used in Blend j.

(a) Formulate this problem as a linear programming model. (14)

Hint: *The objective function can be found by using the formula:*

$$\textbf{\textit{Profit}} = \textbf{\textit{Revenue}} - \textbf{\textit{Cost}}.$$

(b) Write LINGO's codes that will solve the problem formulated in (a) and attach the Solution Report. (3)

(c) Deduce the optimal solution (if it exists). (2)

Question 4 **[21]**

Consider the following LP problem

$$
\begin{aligned}
\text{Minimise} \quad & z = x + 2y \\
\text{subject to} \quad & \\
& -2x + 3y \leq 14 \\
& 5x + 3y \leq 28 \\
& x + 2y \geq 7 \\
& 7x + 2y \geq 7 \\
\text{and} \quad & x, y \geq 0.
\end{aligned}
$$

(a) Represent all the constraints of the problem on a graph. Represent x on the horizontal axis and y on the vertical axis. Label all relevant lines on the graph and indicate the feasible region clearly. Include two isocost lines. (8)
Note: *Use the necessary tools to draw a reasonably accurate graph. A rough sketch is not acceptable.*

(b) Find all corner-points of the feasible region and evaluate the objective function of the LP model at each of them. (4)

(c) Deduce the optimal solution. If the LP problem is infeasible or unbounded, give the reason for this and if the LP problem has multiple optimal solutions, find the general optimal solution. (4)

(d) State clearly all the redundant, binding and nonbinding constraints in this linear programming problem. (3)

(e) Find the general equation of isocost lines. (2)

Question 5 **[25]**

Solve the following LP model.

$$
\begin{aligned}
\text{Maximise} \quad & z = x_1 + 2x_2 + x_3 \\
\text{subject to} \quad & \\
& x_1 + 2x_2 + x_3 = 4 \\
& 3x_1 + x_2 - x_3 = 6 \\
\text{and} \quad & x_1,\ x_2,\ x_3 \geq 0.
\end{aligned}
$$

Note: *If the LP problem is infeasible or unbounded, give the reason for this and if the LP problem has multiple optimal solutions, find the general optimal solution.*

Total: 85 Marks

ASSIGNMENT 3
Only for Semester 1 students
Due date: 26 April 2018
Unique Assignment Number: 746438

Important Notes:

- *Assignment 3 is compulsory and you must submit it to reach Unisa before the due date. It contains only* ***Multi-Choice-Questions*** *and must be submitted via my*Unisa.
- *Assignment 3 consists of 10 questions and covers the entire content of the study guide.*
- *This assignment contributes* ***20%*** *towards your semester mark.*

Question 1

Let the matrix M be given by

$$M = \begin{bmatrix} x & 0 & 1 \\ 0 & 1-x & 0 \\ 1 & 0 & 2-x \end{bmatrix}.$$

Which one of the following statements is false?

[1] The reduced row echelon form of the matrix M is an identity matrix if and only if $x = 1$.

[2] The determinant $|M|$ of the matrix M is $-(1-x)^3$.

[3] The inverse of the matrix M exists if and only if $x \neq 1$ and is given by

$$M^{-1} = \begin{bmatrix} -\dfrac{2-x}{(1-x)^2} & 0 & \dfrac{1}{(1-x)^2} \\ 0 & \dfrac{1}{1-x} & 0 \\ \dfrac{1}{(1-x)^2} & 0 & -\dfrac{x}{(1-x)^2} \end{bmatrix}.$$

[4] The equation $M - 2X = I_3$ is satisfied if and only if

$$X = \frac{1}{2}\begin{bmatrix} -1+x & 0 & 1 \\ 0 & -x & 0 \\ 1 & 0 & 1-x \end{bmatrix}.$$

[5] None of the above.

Question 2

Consider the following system of linear equations

$$\begin{array}{rcrcl} x & + & y & = & 8 \\ x & + & (a^2-15)y & = & 2a \end{array}$$

Which of the following statements is true?

[1] The system has infinitely many solutions if and only if $a = -4$.

[2] The system has no solution for $a = -4$.

[3] The system has a unique solution for all $a \in \mathbb{R}$.

[4] The system has no solution for $a = \pm\sqrt{14}$.

[5] None of the above.

Question 3

Consider the following LP model:

$$\begin{array}{ll} \text{Maximise} & z = 2x - 4y \\ \text{subject to} & \\ & \begin{array}{rcrcr} 2x & - & 4y & \leq & 1 \\ -x & + & 2y & \geq & -2 \\ 3x & - & 5y & \leq & 2 \end{array} \\ \text{and} & \qquad\qquad x, y \geq 0. \end{array}$$

The problem

[1] is degenerate.

[2] is infeasible.

[3] is unbounded and has multiple optimal solutions.

[4] has a unique optimal solution.

[5] None of the above.

Question 4

Consider the following LP model:

$$\begin{array}{lrcl} \text{Maximise} & z = mx + ny & & \\ \text{subject to} & & & \\ & -2x + 3y & = & 6 \\ & 4x + 3y & \leq & 24 \\ & 2x + 3y & \geq & 6 \\ \text{and} & x, y & \geq & 0. \end{array}$$

Which statement is false?

[1] The feasible region of this LP problem is a closed line segment with corner-points given by $(0, 2)$ and $(3, 4)$.

[2] This problem has multiple optimal solutions if $m = -4$ and $n = 6$.

[3] The constraint $2x + 3y \geq 6$ is redundant.

[4] The constraint $4x + 3y \leq 24$ is nonbinding.

[5] None of the above.

Questions 5, 6 and 7 relate to the following situation:

Assume that the following tableau is the final simplex table for an LP problem with the non-negative decision variables x_1, x_2 and x_3 and three constraints. The objective function z is maximized and slack variables s_1 and s_2, and artificial variables a_1 and a_2 were added.

z	x_1	x_2	x_3	s_1	s_2	a_1	a_2	rhs	BV
0	0	0	-1	0	1	1	0	r	a_1
0	0	1	p	1	0	0	1	2	x_2
0	1	0	-4	0	0	0	0	3	s_1
1	6	2	q	9	18	$M-9$	$M+1$	18	z

Question 5

The LP problem has multiple optimal solutions if

[1] $q > 0,\ r \geq 0$ and any p.

[2] $q < 0,\ r = 0$ and any p.

[3] $q = 0,\ r > 0$ and $p < 0$.

[4] $q = 0,\ r = 0$ and $p > 0$.

[5] None of the above.

Question 6

The LP problem is infeasible if

[1] $q < 0,\ r = 0$ and $p < 0$.

[2] $q < 0,\ r = 0$ and $p \geq 0$.

[3] $q > 0,\ r > 0$ and any p.

[4] $q < 0,\ r \geq 0$ and $p \geq 0$.

[5] None of the above.

Question 7

The LP problem is unbounded if

[1] $q < 0,\ r = 0$ and $p > 0$.

[2] $q < 0,\ r = 0$ and $p < 0$.

[3] $q > 0,\ r > 0$ and $p > 0$.

[4] $q = 0,\ r \geq 0$ and $p > 0$.

[5] None of the above.

Questions 8, 9 and 10 relate to the following LP problem:

The Juice Company sells bags of oranges and cartons of orange juice. The company grades oranges on a scale of 1 (poor) to 10 (excellent). At present, the company has on hand 50 000 kg of grade 9 oranges and 60 000 kg of grade 6 oranges. The average quality of oranges sold in bags must be at least 7, and the average quality of the oranges used to produce orange juice must be at least 8.

Each kg of oranges that is used for juice yields a revenue of R30 and incurs a variable cost (consisting of labour costs, variable overhead costs, inventory costs etc.) of R20. Each kg of oranges sold in bags yields a revenue of R10 and incurs a variable cost of R5.

Define the decision variables x_{ij} as the number of kg of grade i oranges used in product j, where $i = 1$ for grade 6, $i = 2$ for grade 9, $j = 1$ for bags and $j = 2$ for juice.

Question 8

The objective function z that maximizes the profit is given by

[1] $10(x_{12} + x_{22}) + 5(x_{11} + x_{21})$.

[2] $10(x_{12} + x_{22}) - 5(x_{11} + x_{21})$.

[3] $30(x_{12} + x_{22}) + 10(x_{11} + x_{21})$.

[4] $30(x_{12} + x_{22}) - 10(x_{11} + x_{21})$.

[5] None of the above.

Question 9

The constraint on the average grade of 7 for oranges sold in the bags is

[1] $\dfrac{x_{11}}{x_{11} + x_{21}} \geq 7$.

[2] $\dfrac{6x_{12} + 9x_{22}}{x_{12} + x_{22}} \geq 7$.

[3] $\dfrac{x_{22}}{x_{12} + x_{22}} \geq 7$.

[4] $\dfrac{6x_{11} + 9x_{21}}{x_{11} + x_{21}} \geq 7$.

[5] None of the above.

Question 10

The availability constraints of both grade 9 and grade 6 oranges are respectively given by

[1] $x_{11} + x_{12} \geq 60\,000$ and $x_{21} + x_{22} \leq 50\,000$.

[2] $x_{21} + x_{22} \leq 60\,000$ and $x_{11} + x_{12} \leq 50\,000$.

[3] $x_{11} + x_{12} = 60\,000$ and $x_{21} + x_{22} = 50\,000$.

[4] $6x_{11} + 9x_{12} \leq 60\,000$ and $6x_{21} + 9x_{22} \leq 50\,000$.

[5] None of the above.

6.2 Second Semester Assignments

ASSIGNMENT 1
Only for Semester 2 students
Due date: 3 August 2018
Unique Assignment Number: 815996

Important Notes:

- ♢ ***Assignment 1 is compulsory and you must submit it to reach UNISA before the due date.*** *No late assignments will be marked.*
- ♢ *Admission to the examination shall be obtained with this assignment.*
- ♢ ***Assignment 1 consists of 5 questions and covers learning units 1 to 4 of the study guide.***
- ♢ *This assignment contributes* ***20%*** *towards your semester mark.*

Question 1: Discussion on *my*Unisa **[7]**

Marks will be awarded for your participation and contribution on the Discussion Forum on *my*Unisa by the due date. You will thus need to visit the module site on *my*Unisa at least one hour per week.

(a) Write down topics that you have created under a specific discussion forum. (2)

(b) State clearly all contributions in different topics created either by you, your fellow students or your lecturers. (5)

Question 2 **[42]**

Consider the following matrices:

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & -1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1 & 1 \\ p^2 & q^2 & r^2 \\ p & q & r \end{bmatrix},$$

where p, q and r are real parameters.

(a) Find the inverse of the matrix A by using the Gauss–Jordan method. (10)

(b) Prove that the determinant $|B| = (r-q)(r-p)(p-q)$. (6)

Hint: The matrix B is a variant of "*Vandermonde matrix*". Use different properties of determinants to find $|B|$.

(c) Assume that X is an unknown square matrix of order 3. For all $p \neq q \neq r$, find all entries of the matrix X such that $A - BX = -2I_3$. Here I_3 represents the identity matrix of order 3. (20)

Hint: Solve the equation in matrix form and find all entries of the matrix X. The inverse B^{-1} may be computed by using the cofactor method with the determinant $|B|$ given in (b).

(d) Find an equivalent matrix B' obtained by performing the following series of elementary row operations on the matrix B. (6)

$$\begin{aligned} R_2' &= \frac{1}{r^2}R_2 \\ R_1' &= R_1 - r^2R_2' \\ R_3' &= R_3 - rR_2'. \end{aligned}$$

Question 3 [18]

Consider the following system of linear equations.

$$\begin{array}{rcrcrcr} -x & + & 4y & + & 2z & = & -2 \\ x & - & 2y & - & 3z & = & 1 \\ 2x & + & y & & & = & 4 \\ 2x & + & 3y & - & z & = & 3 \end{array}$$

(a) List all leading and free variables while solving the system by using the Gauss-Jordan elimination method.

(b) Solve the linear system above and write down the solution set clearly.

Question 4 **[13]**

Consider the following system of linear equations.

$$\begin{array}{rcrcrcl} x & + & y & + & 3z & = & a \\ x & + & 2y & & & = & a \\ x & + & y & + & (a^2-1)z & = & a \end{array}$$

Find all values of the parameter a for which the linear system above has :

(a) no solution.

(b) a unique solution. Which one ?

(c) infinitely many solutions. Find the general solution.

Question 5 **[10]**

Consider the following matrices:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}, \ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Find a nontrivial solution $\mathbf{x}$ to the homogeneous system

$$(I_3 - 2A)\mathbf{x} = \mathbf{0}.$$

Total: 90 Marks

ASSIGNMENT 2
Only for Semester 2 students
Due date: 7 September 2018
Unique Assignment Number: 821150

Important Notes:

- ◇ **Assignment 2 is compulsory and you must submit it to reach Unisa before the due date.** ***No late assignments will be marked.***
- ◇ Assignment 2 consists of 5 questions and covers learning units 5 to 9 of the study guide.
- ◇ This assignment contributes **60%** towards your semester mark.

Question 1 [7]

Marks will be awarded for your participation and contribution on the Discussion Forum on *my*Unisa by the due date. You will thus need to visit the module site on *my*Unisa at least one hour per week.

(a) Write down topics that you have created under a specific discussion forum. (2)

(b) State clearly all contributions in different topics created either by you, your fellow students or your lecturers. (5)

Question 2 [15]

A bank makes four kinds of loans to its personal customers and these loans yield the following annual interest rates to the bank:

- First mortgage 12.5%
- Second mortgage 14%
- Home improvement 18%
- Personal overdraft 8%

The bank has a maximum foreseeable lending capability of 800 million rands and is further constrained by the following policies:

1. first mortgages must be at least 40% of all mortgages issued and at least 20% of all loans issued (in rands terms)
2. second mortgages cannot exceed 45% of all loans issued (in terms of rands)

3. to avoid public displeasure and the introduction of a new windfall tax the average interest rate on all loans must not exceed 15%.

(a) Define all decision variables of the problem. (2)

(b) Formulate the bank's loan problem as an LP that maximises interest income whilst satisfying the policy limitations. (7)
Note: *These policy conditions, whilst potentially limiting the profit that the bank can make, also limit its exposure to risk in a particular area. It is a fundamental principle of risk reduction that risk is reduced by spreading money (appropriately) across different areas.*

(c) Write LINGO's codes that will solve the problem formulated in (b) (2)

(d) Deduce the optimal solution (if it exists) using LINGO and attach the Solution Report. (4)

Question 3 [18]

An oil company has three different processes that can be used to manufacture various types of petrol. Each process involves blending oils in the company's catalytic cracker.

– Running Process 1 for an hour costs R100 and requires 2 barrels of crude oil 1 and 3 barrels of crude oil 2. The output from running Process 1 for an hour is 2 barrels of Petrol 1 and 1 barrel of Petrol 2.

– Running Process 2 for an hour costs R80 and requires 1 barrel of crude oil 1 and 3 barrels of crude oil 2. The output from Process 2 for an hour is 3 barrels of Petrol 2.

– Running Process 3 for an hour costs R20 and requires 2 barrels of crude oil 2 and 3 barrels of Petrol 2. The output from running Process 3 for an hour is 2 barrels of Petrol 3.

Each week, 200 barrels of crude oil 1, at R40 per barrel, and 300 barrels of crude oil 2 at R60 per barrel, may be purchased.
All petrol produced can be sold at the following per-barrel prices: Petrol 1, R180; Petrol 2, R200; Petrol 3, R480. Assume that only 100 hours of time on the catalytic cracker are available each week.

(a) Define all decision variables of the problem. (2)

(b) Formulate an LP whose solution will maximize revenues less costs. (10)

Note: *The objective function of this problem is a bit tricky to find. We would recommend you to use the following formula:*

$$\textbf{\textit{Objective Function}} = \textbf{\textit{Total Revenue}}\ (TR) - \textbf{\textit{Total Cost}}\ (TC),$$

where

TR = *Revenue from Petrol 1 + Revenue from Petrol 2 + Revenue from Petrol 3*

and where

TC = *Cost for running processes 1,2 and 3 + Cost for oil.*

(c) Write LINGO's codes that will solve the problem formulated in (b). (2)

(d) Deduce the optimal solution (if it exists) using LINGO and attach the Solution Report. (4)

Question 4 [20]

Consider the following LP problem

$$\begin{array}{ll} \text{Maximise} & z = -12x + 17y \\ \text{subject to} & \\ & \begin{array}{rcl} 8x + 3y & \geq & 24 \\ 5x - 2y & \leq & 30 \\ -12x + 17y & \leq & 50 \\ y & \geq & 2 \end{array} \\ \text{and} & \quad x, y \geq 0. \end{array}$$

(a) Represent all the constraints of the model on a graph. Represent x on the horizontal axis and y on the vertical axis. Label all relevant lines on the graph and indicate the feasible region clearly. Include two isoprofit lines on the graph. (9)

Note: *Use necessary tools to draw a reasonably accurate graph. A rough sketch is not acceptable.*

(b) Find all corner-points of the feasible region and evaluate the objective function at each of them. (4)

(c) Deduce the optimal solution. If the LP problem has multiple optimal solutions, then find the general optimal solution. (3)

(d) State clearly all the redundant, binding and nonbinding constraints in this linear programming problem. (3)

(e) Find the general equation of isoprofit lines. (2)

Question 5 **[25]**

Solve the LP problems below.

$$
\begin{array}{lrcrcrcr}
\text{Maximise} & z = x_1 + 4x_2 + 2x_3 & & & & & & \\
\text{subject to} & & & & & & & \\
& 4x_1 & + & x_2 & + & 2x_3 & \geq & 5 \\
& -x_1 & + & x_2 & + & 2x_3 & \leq & 10 \\
& x_1 & + & 4x_2 & + & 2x_3 & \leq & 2 \\
\text{and} & x_1,\ x_2,\ x_3 \geq 0. & & & & & &
\end{array}
$$

Note: *If the LP problem is infeasible or unbounded, give the reason for this and if the LP problem has multiple optimal solutions, find the general optimal solution.*

Total: 85 Marks

ASSIGNMENT 3
Only for Semester 2 students
Due date: 21 September 2018
Unique Assignment Number: 889072

Important Notes:

- ♢ *Assignment 3 is compulsory and you must submit it to reach Unisa before the due date. It contains only* ***Multi-Choice-Questions*** *and must be submitted via* my*Unisa.*
- ♢ *Assignment 3 consists of 10 questions and covers the entire content of MO001.*
- ♢ *This assignment contributes* ***20%*** *towards your semester mark.*

Question 1

Let the matrix M be given by

$$M = \begin{bmatrix} 1+k & 0 & 0 \\ 1 & (1+k)^2 & 0 \\ 1 & 1 & (1+k)^3 \end{bmatrix}.$$

Which one of the following statements is false?

[1] The equation $3M - 2X = I_3$ is satisfied if and only if

$$X = \frac{1}{2}\begin{bmatrix} 1-3(1+k) & 0 & 0 \\ -3 & 1-3(1+k)^2 & 0 \\ -3 & -3 & 1-3(1+k)^3 \end{bmatrix}.$$

[2] The determinant $|M|$ of the matrix M is $(1+k)^6$.

[3] The inverse of the matrix M exists if and only if $k \neq -1$ and is given by

$$M^{-1} = \begin{bmatrix} \frac{1}{(1+k)} & 0 & 0 \\ -\frac{1}{(1+k)^3} & \frac{1}{(1+k)^2} & 0 \\ -\frac{k(2+k)}{(1+k)^6} & -\frac{1}{(1+k)^5} & \frac{1}{(1+k)^3} \end{bmatrix}.$$

[4] The reduced row echelon form of the matrix M is the identity matrix I_3 if and only if $k \neq -1$.

[5] None of the above.

Questions 2 and 3 relate to the following system of linear equations:
Consider the following system of linear equations

$$\begin{array}{ccccccccc} w & + & x & + & y & + & z & = & 6 \\ w & & & + & y & + & z & = & 4 \\ w & & & + & y & & & = & 2. \end{array}$$

Question 2

In the process of Gauss-Jordan elimination method, the leading and free variables of the system are given respectively by

[1] Leading variables are w, y and z and the free variable is x.

[2] Leading variables are w, x and z and the free variable is y.

[3] Leading variables are w, x and y and the free variable is z.

[4] Leading variables are y and z and the free variables are w and x.

[5] None of the above.

Question 3

The general solution of the linear system is

[1] $w = t, \quad x = 2, \quad y = 2 - t$ and $z = 2 - t, \quad$ for all $t \in \mathbb{R}$.

[2] $w = 2, \quad x = 2, \quad y = t$ and $z = 2, \quad$ for all $t \in \mathbb{R}$.

[3] $w = 2 - t, \quad x = 2, \quad y = t$ and $z = 2, \quad$ for all $t \in \mathbb{R}$.

[4] $w = 2, \quad x = 2, \quad y = 0$ and $z = 2$.

[5] None of the above.

Question 4

Consider the following LP model:

$$\begin{array}{lrcl}
\text{Maximise} & z = -x + 4y & & \\
\text{subject to} & & & \\
 & 3x + 4y & \geq & 24 \\
 & -3x + 4y & \leq & 12 \\
 & x & \leq & 6 \\
 & 2y & \leq & 15 \\
\text{and} & x, y & \geq & 0.
\end{array}$$

Which statement is false?

[1] The feasible region of this LP problem exists with corner-points given by $(2, \frac{9}{2})$, $(6, \frac{3}{2})$ and $(6, \frac{15}{2})$.

[2] The isoprofit lines satisfy the equation $4y - x = b$, where $b \in \mathbb{R}$.

[3] The constraint $2y \leq 15$ is redundant.

[4] The optimal value of the objective function occurs at the corner-point $(2, \frac{9}{2})$ and is given by $z = 16$.

[5] None of the above.

Questions 5, 6 and 7 relate to the following linear programming problem:

A furniture company manufactures tables and chairs. A table requires 40 square units of wood, and a chair requires 30 square units of wood. Wood may be purchased at a cost of R10 per square unit, and 40 000 square units of wood are available for purchase. It takes two hours of skilled labour to manufacture an unfinished table or an unfinished chair. Three more hours of skilled labour will turn an unfinished table into a finished table, and two more hours of skilled labour will turn an unfinished chair into a finished chair. A total of 6 000 hours of skilled labour are available (and have already been paid for). All furniture produced can be sold at the following unit prices:

- *unfinished table R700*
- *finished table R1400*
- *unfinished chair R600*
- *finished chair R1100*

Define x_i, $(i = 1, 2, 3, 4)$ as the decision variables, where x_1 represents the number of unfinished tables produced, x_2 represents the number of finished tables produced, x_3 represents the number of unfinished chairs produced and x_4 represents the number of finished chairs produced.

Question 5

The objective function z that will maximise the contribution to profit from manufacturing tables and chairs is given by

[1] $-700x_1 + 1\,400x_2 - 600x_3 + 1\,100x_4$.

[2] $700x_1 + 1\,400x_2 + 600x_3 + 1\,100x_4$.

[3] $300x_1 - 1\,000x_2 + 300x_3 + 800x_4$.

[4] $660x_1 + 1\,360x_2 + 570x_3 + 1\,070x_4$.

[5] None of the above.

Question 6

The constraint on wood used is

[1] $40x_1 + 40x_2 + 30x_3 + 30x_4 = 40\,000$.

[2] $40x_1 + 30x_2 + 40x_3 + 30x_4 \leq 40\,000$.

[3] $40x_1 + 30x_2 + 40x_3 + 30x_4 = 40\,000$.

[4] $40x_1 + 40x_2 + 30x_3 + 30x_4 \leq 40\,000$.

[5] None of the above.

Question 7

The constraint on skilled labour is

[1] $2x_1 + 2x_2 + 2x_3 + 4x_4 \leq 6\,000$.

[2] $2x_1 + 3x_2 + 2x_3 + 2x_4 \leq 6\,000$.

[3] $2x_1 + 5x_2 + 2x_3 + 4x_4 \leq 6\,000$.

[4] $2x_1 + 3x_2 + 2x_3 + 4x_4 \leq 6\,000$.

[5] None of the above.

Questions 8, 9 and 10 relate to the following simplex tableau:

Consider the following simplex tableau for an LP problem with non-negative decision variables x_1, x_2 and x_3 and three constraints. The objective function z is maximized and slack variables s_1 and s_2, and artificial variables a_1 and a_2 were added.

z	x_1	x_2	x_3	s_1	s_2	a_1	a_2	rhs	BV
0	0	0	-1	-1	-5	1	0	5	x_2
0	0	1	-2	0	-1	0	1	0	s_2
0	1	0	q	1	2	2	0	8	s_1
1	2	0	p	0	2	$M+1$	$2M-9$	18	z

Question 8

If $p < 0$ and $q < 0$, then the solution to the LP problem is:

[1] $z = 18,\ x_1 = 0,\ x_2 = 5$ and $x_3 = 0$.

[2] unbounded.

[3] infeasible.

[4] $z = 18,\ x_1 = 5,\ x_2 = 0$ and $x_3 = 8$.

[5] None of the above.

Question 9

If $p > 0$ and $q < 0$, then the solution to the LP problem is:

[1] $z = 18,\ x_1 = 0,\ x_2 = 5$ and $x_3 = 0$.

[2] unbounded.

[3] infeasible.

[4] $z = 18,\ x_1 = 5,\ x_2 = 0$ and $x_3 = 8$.

[5] None of the above.

Question 10

If $p = 0$ and $q = 2$, then

[1] the problem has a unique optimal solution given by $z = 18$, $x_1 = 0$, $x_2 = 5$ and $x_3 = 0$.

[2] the problem has a unique optimal solution given by $z = 18$, $x_1 = 0$, $x_2 = 9$ and $x_3 = 4$.

[3] the problem has multiple optimal solutions where the general optimal solution is given by $z = 18$ and

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \alpha \begin{bmatrix} 0 \\ 9 \\ 4 \end{bmatrix} + (1-\alpha) \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix},$$

for all $\alpha \in [0, 1]$.

[4] the problem has multiple optimal solutions where the general optimal solution is given by $z = 18$ and

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \alpha \begin{bmatrix} 5 \\ 0 \\ 8 \end{bmatrix} + (1-\alpha) \begin{bmatrix} 9 \\ 8 \\ 4 \end{bmatrix},$$

for all $\alpha \in [0, 1]$.

[5] None of the above.

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