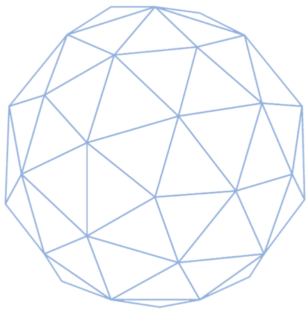
Elementary Mathematics

MO001 FOR
QMI1500

Department of Decision Sciences

UNIVERSITY OF SOUTH AFRICA
PRETORIA



Printed and distributed by the
University of South Africa,
Muckleneuk, Pretoria.

QMI1500/MO001/4/2017

70495750

ELEMENTARY QUANTITATIVE METHODS

The purpose of this module is to enable a student to become conversant with a variety of basic quantitative techniques

Contents

COMPONENT 1

Numbers and working with numbers

On completion of this component you should have obtained basic numeracy skills.

CONTENTS

Study unit 1.1 Priorities and laws of operations

Learning objectives: *On completion of this study unit you should*

- *know and be able to apply the priority rules in solving problems*
- *know and be able to apply the laws of operations in solving problems*

1.1.1 Priorities

What are the values of the expressions

$$2 \times 3 + 4 \times 5 \text{ and } 8 \div 2 - 6 \div 3?$$

I hope you are able to answer 26 and 2 respectively, without using your calculator. How do you derive these results? In the first case, 2×3 is equal to 6, 4×5 is 20 and $6 + 20$ is 26. In the second case $8 \div 2$ is equal to 4, $6 \div 3$ is 2 and $4 - 2$ is 2. Elementary!

Yes. It is elementary, but why?

It is essential that when we see an expression we all understand the same thing and obtain the same result or else all our numbers and operations will lead to nonsense. Therefore, we all agree to perform the operations in a specific order.

Looking at the above two expressions and their accepted results, it is evident that multiplication takes precedence over addition, and division takes precedence over subtraction. But what about more complex expressions like

$$2 \times 3 - 4 \times 5 + 6 \div 3?$$

Here, too, multiplication and division take precedence over addition and subtraction, but whether you do the multiplication before the division and the subtraction before the addition, or vice versa, does not matter. Check for yourself.

The commonly accepted **convention** is thus: multiplication and division have a higher priority than addition and subtraction.

You may wonder whether there are operations that have a higher priority than multiplication and division. Yes, there are. Specifically, the operation "change sign", which changes a positive number to a negative number and vice versa, has the highest priority of all. Next comes exponentiation in its various forms. The accepted priority level is the following:

Highest priority
Change sign
Exponentiation in all its forms
Multiplication and division
Addition and subtraction
Lowest priority

When you have more than one operation with the same priority, you just operate from left to right. For instance,

$$15 \div 3 \times 4$$

is not $15 \div 12$, but is rather 5×4. This gives

$$5 \times 4 = 20$$

because, going from left to right, you get to the division first.

There are exceptions, however, for example $-2^{\frac{1}{2}}$, where division takes priority over exponentiation, which takes priority over change of sign. This will be dealt with in study unit 1.4.

Activity

Calculate each of the following expressions. In each case state the order in which the operations are performed, assuming that you operate from left to right when priorities are equal:

1. $4 \times 3 \div 2 + 5 \times 6 - 20 \div 4 \times 2$
2. $3^2 \times 7 - \sqrt{9} \times 8 + 2^2 \div 4 \times 3$

Answer

Simplifying the expressions gives the following:

1.
$$\begin{aligned} 4 \times 3 \div 2 + 5 \times 6 - 20 \div 4 \times 2 &= 12 \div 2 + 30 - 5 \times 2 \\ &= 6 + 30 - 10 \\ &= 26 \end{aligned}$$

2.
$$\begin{aligned} 3^2 \times 7 - \sqrt{9} \times 8 + 2^2 \div 4 \times 3 &= 9 \times 7 - 3 \times 8 + 1 \times 3 \\ &= 63 - 24 + 3 \\ &= 42 \end{aligned}$$

1.1.2 Brackets

Say, for instance, that in the example

$$2 \times 3 + 4 \times 5$$

you actually want to add 3 and 4 first and then multiply by 2 and 5. How can we achieve this? The answer, as you are perhaps aware, since we have already resorted to their use from time to time, is brackets. We therefore write

$$2 \times (3 + 4) \times 5$$

and understand that the expression in brackets, namely $(3+4)$, must be calculated first and then the other operations. In this case we thus obtain, $3 + 4$ is 7, 2×7 is 14 and 14×5 is 70.

Another example is the use of brackets to indicate fractional exponents, for example

$$8^{\frac{2}{3}} \text{ is } 8^{(2 \div 3)}.$$

Brackets are used to override the preconceived priorities.

It is at this stage, I believe, evident that brackets are a very powerful means of casting expressions in a form that conveys the specific meaning that we want. The point is that any expression between brackets is regarded as a number which must be determined first before proceeding further with the calculation.

"Any expression?" But an expression itself may contain brackets. Does this mean that we can have brackets within brackets? Yes, indeed it does. That is, in fact, the beauty and the power of brackets. Let's look at a more complex example.

Consider

$$\sqrt{\left(3 \times \sqrt{(4^2+3^2)} + (4+3)^2\right)}.$$

This means that you take the square root of the expression between the outermost brackets. However, this cannot be done until we know the value of the expression contained by the innermost brackets. Thus, the innermost brackets are calculated first. Note that within a set of brackets the usual priority rules apply, for example exponentiation before addition in the case of the first inner bracket. Step by step, the calculation will take place as follows:

Step	**Reduced expression**
(1)	$\sqrt{\left(3 \times \sqrt{(4^2+3^2)} + (4+3)^2\right)}$
(2)	$\sqrt{\left(3 \times \sqrt{25} + 7^2\right)}$
(3)	$\sqrt{(3 \times 5 + 49)}$
(4)	$\sqrt{(15 + 49)}$
(5)	$\sqrt{64}$
(6)	8

Whether we evaluate the first or second inner bracket first does not matter.

Activity

Set out the steps which will be executed in the calculation of the following expression and check your result with your calculator:

$$\left(4^2 - (6 \times 3 - 80 \div 5)^2\right) \div \left(9^2 - 77\right)$$

Answer

The answer is as follows:

$$\begin{aligned} & \left(4^2 - (6 \times 3 - 80 \div 5)^2\right) \div \left(9^2 - 77\right) \\ = \; & \left(4^2 - (18 - 16)^2\right) \div \left(9^2 - 77\right) \\ = \; & \left(4^2 - 2^2\right) \div \left(9^2 - 77\right) \\ = \; & (16 - 4) \div (81 - 77) \\ = \; & 12 \div 4 \\ = \; & 3 \end{aligned}$$

This exercise illustrates, in some detail, the fact that any expression between brackets is simply a number which has to be determined before proceeding further.

For every left bracket in an expression there must be a corresponding right bracket and vice versa, otherwise the expression is not uniquely specified. Always work from the innermost bracket outwards.

Lastly, a useful hint about brackets and priorities. If you are unsure about the relative priorities of different operations, use brackets to enforce the priority order you want. A set or two of extra brackets, even if they are redundant, often do much to enhance the readability of an expression.

1.1.3 Laws of operations

There are specific laws that apply to combinations of the basic operations.

The two most basic laws are illustrated in the following results:

For addition:

$$6 + 2 = 2 + 6 \qquad \text{(and both are equal to 8)}$$

or

$$3{,}1 + 9{,}3 = 9{,}3 + 3{,}1 \qquad (= 12{,}4)$$

or

$$-7{,}6 + 2{,}3 = 2{,}3 + -7{,}6 \qquad (= -5{,}3)$$

or

$$-11{,}1 + -4{,}4 = -4{,}4 + -11{,}1 \qquad (= -15{,}5)$$

For multiplication:

$$5 \times 2 = 2 \times 5 \qquad (= 10)$$

or

$$2{,}2 \times 3{,}5 = 3{,}5 \times 2{,}2 \qquad (= 7{,}7)$$

or

$$-8{,}1 \times 9{,}4 = 9{,}4 \times -8{,}1 \qquad (= -76{,}14)$$

or

$$-11 \times -13 = -13 \times -11 \qquad (= 143)$$

The two laws involved are the **commutative laws of addition and multiplication**.

Commutative law of addition:

The sum of two numbers is unique, that is, it does not matter which number is placed first and which second, the result is the same.

Commutative law of multiplication:

The product of two numbers is unique.

What about subtraction and division? Does the commutative law apply to them too? A counter example shows us that it does not, for example

$$7 - 3 \neq 3 - 7 \quad \text{and} \quad 10 \div 2 \neq 2 \div 10.$$

The sign $\neq$ is read as "does not equal". In the first case, subtraction, commutation (ie changing the order) leads to the negative of the initial result (ie -4 instead of 4); whereas in the second case, division, commutation leads to the inverse of the initial result (ie $1 \div 5$ instead of 5).

The conclusion is that, whereas for addition and multiplication we can change the order of the factors, for subtraction and division we cannot, or rather, if we do, we must do so with care.

The next two laws apply to addition and multiplication. Consider the following, where we use brackets to indicate the sequence in which the operations are to be performed.

For addition:

$$7 + (3 + 2) = (7 + 3) + 2 \qquad (= 12)$$

$$6{,}1 + (-5{,}1 + 3{,}7) = (6{,}1 + -5{,}1) + 3{,}7 \qquad (= 4{,}7)$$

$$-9{,}3 + (2{,}2 + 4{,}5) = (-9{,}3 + 2{,}2) + 4{,}5 \qquad (= -2{,}6)$$

For multiplication:

$$
\begin{aligned}
5 \times (4 \times 2) &= (5 \times 4) \times 2 && (= 40) \\
6{,}1 \times (7{,}3 \times 5{,}5) &= (6{,}1 \times 7{,}3) \times 5{,}5 && (= 244{,}915) \\
-9 \times (5 \times -7) &= (-9 \times 5) \times -7 && (= 315)
\end{aligned}
$$

The two laws involved are the **associative laws of addition and multiplication**.

Associative law of addition:

> The sum of three numbers does not depend on which two are added first – the result is unique.

Associative law of multiplication:

> The product of three numbers does not depend on which two are multiplied first – the result is unique.

This implies, for example, that when adding long columns of numbers, it does not matter which numbers we add first and which last. A similar remark applies to multiplication, but when it comes to subtraction and division, we must be very careful, as the following examples show:

$$10 - (5 - 2) \neq (10 - 5) - 2$$

and

$$(8 \div 4) \div 2 \neq 8 \div (4 \div 2)$$

This means that an expression like $8 \div 4 \div 2$ is highly ambiguous because the result depends on the order in which the operations are performed. It is therefore preferable that you indicate and enforce a specific order using brackets.

So far, we have stated laws for expressions containing one type of operation only – either addition or multiplication. Let us take a look at combinations of these two operations.

Activity

Confirm that the expressions to the left and the right of the equal sign yield the same results:

1. $6 \times (2 + 3) = 6 \times 2 + 6 \times 3$
2. $7{,}2 \times (2{,}2 + 3{,}3) = 7{,}2 \times 2{,}2 + 7{,}2 \times 3{,}3$
3. $-5 \times (-4 + 6) = -5 \times -4 + -5 \times 6$
4. $12{,}3 \times (-3{,}4 + -6{,}6) = 12{,}3 \times -3{,}4 + 12{,}3 \times -6{,}6$

What we observe in this activity is an illustration of the following law:

Distributive law of multiplication over addition:

> The product of a number with the sum of two other numbers is equal to the sum of the products of the first number with each of the other two numbers.

We can take this one step further. Since multiplication obeys the commutative law, the factors involved in the above statement may be reversed. Thus, for example:

$$\begin{aligned} 6 \times (2+3) &= 6 \times 2 + 6 \times 3 \\ &= 2 \times 6 + 3 \times 6 \\ &= (2+3) \times 6 \end{aligned}$$

In other words, it does not matter whether we multiply from the left or the right in expressions of this type. Similarly whether we write $2 + 3$ or $3 + 2$, the result is the same since the commutative law of addition applies.

Furthermore, although we cannot easily formulate similar laws for subtraction and division, we can frequently transform expressions containing subtraction and division to ones containing only addition and multiplication. To be specific, in the case of subtraction we change the sign of the number being subtracted and then add. For example:

$$\begin{aligned} 11 - 4 &= 11 + -4 \\ &= -4 + 11, \end{aligned}$$

where we use the commutative law in the last step.

In the case of division we can replace the number being divided by, by its inverse and multiply, for example:

$$\begin{aligned} 9 \div 3 &= 9 \times 3^{-1} \\ &= 3^{-1} \times 9 \end{aligned}$$

again using the commutative law in the last step. Or, consider an application of the distributive law:

$$\begin{aligned} (17-4) \div 5 &= (17 + -4) \times 5^{-1} \\ &= 17 \times 5^{-1} + -4 \times 5^{-1} \end{aligned}$$

Although each law is elementary almost to the extent of being self-evident, used in conjunction with each other the three laws can be very powerful tools for rearranging expressions. We shall have ample opportunity to make good use of them later.

Exercise 1.1

1. Apply the distributive law of multiplication over addition to expand the following expression to one containing the sum of four terms, each term being the product of three numbers. Do not actually calculate the result.

   The expression is $7 \times (6 \times (5+4) + 3 \times (2+1))$.

2. Which of the commutative or associative laws are associated with the following expressions?

   (a) $2 + (5 + 4) = (2 + 5) + 4$
   (b) $(3 + 7) + 4 = (7 + 3) + 4$
   (c) $(7 \times 5) \times 2 = 2 \times (7 \times 5)$
   (d) $2 \times (7 \times 4) = (4 \times 2) \times 7$

Study unit 1.2 Variables

Learning objectives: *On completion of this study unit you should be able to express relations between numbers in general terms by making use of symbols or letters.*

If you want to explain to someone how to find the area of a rectangle 4 cm by 3 cm, you can tell them to multiply 4 by 3. However, if you do not have specific values for the rectangle, you can call the length of the rectangle L and its width W, then you can say that the area is $L \times W$ – this will be true for any value of L and W. Consider the rule for expressing one quantity as a percentage of another: "Put the first quantity on top of the second and multiply by 100." What a mouthful! However, if the first number is denoted by x and the second by y, then the rule boils down to

$$\frac{x}{y} \times 100.$$

This will be true for any values of x and y. This is the great strength of using letters or symbols to represent numbers – we are then able to write down rules, expressions and so on that are completely general, so that to find the answer in a particular case all we need to do is substitute our particular values of x and y, or whatever, into the appropriate expression.

The main purpose of using letters or symbols to represent numbers or quantities is that it enables us to express practical truths about the real world neatly, succinctly and in more general terms than we could if we insisted on sticking to definite numbers all the time.

This is illustrated with the laws discussed previously.

Consider the **Commutative law of addition:**

> The sum of two numbers is unique, that is it does not matter which number is placed first and which second, the result is the same.

Suppose we use letters A and B. Then this law can be written as

$$A + B = B + A.$$

By using the symbols or letters A, B and C, the various laws can be succinctly stated as follows:

Commutative law of addition

$$A + B = B + A$$

Commutative law of multiplication

$$A \times B = B \times A$$

Associative law of addition

$$(A + B) + C = A + (B + C)$$

Associative law of multiplication

$$(A \times B) \times C = A \times (B \times C)$$

Distributive law of multiplication over addition

$$A \times (B + C) = A \times B + A \times C.$$

It would surprise me if you did not agree that, in general, these statements are more concise and clear than those of the previous study unit.

Four other fundamental rules which were dealt with in passing were the following:

The product of any number with zero is zero.
The product of any number with 1 is the number.
The sum of any number and its negative is zero.
The product of any number and its inverse is 1.

In terms of letters or symbols, these are elegantly stated:

The product of any number with zero is zero:

$$A \times 0 = 0$$

The product of any number with 1 is the number:

$$A \times 1 = A$$

The sum of any number and its negative is zero:

$$A + -A = 0$$

The product of any number and its inverse is 1:

$$A \times A^{-1} = 1$$

Again, we could have used any one of the symbols A to Z to formulate these two laws.

Everyday life abounds with examples of variables. A firm's gross monthly sales vary from month to month, usually in number and value. Your telephone account or water and lights account is unlikely to be the same each month. Nor is the rainfall or the average maximum temperature from day to day. Even seemingly constant items such as your flat or house rent (or your salary!) are adjusted periodically and are therefore variables.

A **variable** is something which can assume any one of several numeric values.

The symbols most often used to represent variables are letters of the alphabet, both upper case such as A, B, C, ... and lower case such as x, y, z.

The **symbol** or letter which is used to represent a variable is simply a label. Its specific value has to be assigned directly or determined by calculation in each relevant instance. Always choose a symbol that is short and that makes sense.

Exercise 1.2

1. Determine the numerical value of the following:
   (a) $12x + 17$ if $x = 2$
   (b) $x^2 - 3$ if $x = 4$
   (c) $2x^2$ if $x = 3$
   (d) $4x - 1$ if $x = 5$
   (e) $\frac{x+7}{4} + 3$ if $x = 5$
   (f) $(x+3)(x-2)$ if $x = 6$
   (g) $10 - 3x$ if $x = 1$
   (h) $\frac{x}{2} + \frac{x}{3}$ if $x = 12$
2. Substitute x with 3 and calculate:
   (a) $5x + 7$
   (b) $x + 3x - 1$
   (c) $5x^2 - 9$
   (d) $\frac{x+4}{7}$
   (e) $2(x+4)$
   (f) $7 - x + 2$
3. If $a = 2$, $b = 1$ and $c = 7$, then determine the value of the following:
   (a) $2(a + b - c) + c(b - a)$
   (b) $a^2 + b^2 + c^2$
   (c) $(a + b)(b - c)$
   (d) $2b - \frac{c+3}{a} + b^2$
4. Write the following as a mathematical expression:
   (a) the sum of x and y
   (b) subtract the sum of a and b from 8
   (c) three times x added to two times y
   (d) Robert's age in seven years' time if he is now y years old

Study unit 1.3 Fractions

Learning objectives: *On completion of this study unit you should be able to do all the basic mathematical operations on fractions.*

1.3.1 Division and multiplication of fractions

The fraction $\frac{10}{2}$ means that ten must be split into a number of groups of size two each. The question is, what is the number of groups? The answer is

$$2 + 2 + 2 + 2 + 2 = 10.$$

Five groups of size two each, added together, will give ten. In short,

$$\frac{10}{2} = 5.$$

What about 4 divided by $\frac{1}{2}$?

In mathematical notation this is $\frac{4}{\frac{1}{2}}$, which is the same as $4 \div \frac{1}{2}$ or $\frac{4}{1} \div \frac{1}{2}$.

The $\frac{1}{2}$ poses a problem. How can one get rid of it? Maybe it could be changed to "something" that is easier to work with.

To make the $\frac{1}{2}$ more manageable, we multiply it by $\frac{2}{1}$, that is

$$\frac{1}{2} \times \frac{2}{1} = \frac{2}{2} = 1.$$

To multiply fractions, we multiply the numerators by each other and the denominators by each other. The **numerator** is the **top** part of a fraction and the **denominator** is the **bottom** part of a fraction.

Take the rule that whatever is done on the left-hand side of an equation also has to be done on the right-hand side, then expand it to

> whatever is done to the *top* part of a fraction has to be done to the *bottom* part as well.

Thus:

$$\begin{aligned} \frac{\frac{4}{1}}{\frac{1}{2}} \times \frac{\frac{2}{1}}{\frac{2}{1}} &= \frac{4 \times 2}{1} \\ &= 4 \times 2 \\ &= 8. \end{aligned}$$

(Note that $\frac{\frac{2}{1}}{\frac{2}{1}} = 1$; that means we do not change our original problem, we multiply it by 1 in a manner that suits us).

Now that we have the idea, let's find some short cuts!

Take the bottom part of the fraction and turn it around, using division's brother, the multiplication sign, as accomplice. You then get:

$$\frac{4}{\frac{1}{2}} = 4 \div \frac{1}{2} = 4 \times \frac{2}{1} = 8,$$

which is exactly the same answer!

Let us try another one: $\dfrac{15}{\frac{1}{3}}$

$$\frac{15}{\frac{1}{3}} = \frac{\frac{15}{1}}{\frac{1}{3}} \times \frac{\frac{3}{1}}{\frac{3}{1}} = \frac{\frac{45}{1}}{\frac{3}{3}} = 45$$

or

$$\frac{15}{\frac{1}{3}} = 15 \times \frac{3}{1} = 45$$

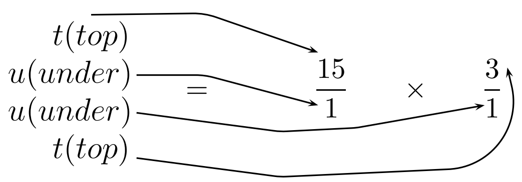
(quicker and easier!).

The 15 can be written as $\dfrac{15}{1}$ and therefore

$$\frac{15}{\frac{1}{3}} = \frac{\frac{15}{1}}{\frac{1}{3}} = \frac{15}{1} \times \frac{3}{1}.$$

Let us take a closer look: the $\dfrac{15}{1}$ stays as it is but the $\dfrac{1}{3}$ changed to $\dfrac{3}{1}$. If we take $\dfrac{\frac{15}{1}}{\frac{1}{3}}$ and attach the following letters, we can write down the answer immediately.

$$\frac{\frac{15}{1}}{\frac{1}{3}}$$



Get the idea?

The following fraction should then not be so difficult:

$$\frac{\frac{3}{4}}{\frac{5}{6}} \quad \begin{matrix} t \\ u \\ u \\ t \end{matrix} \quad = \quad \frac{3}{4} \times \frac{6}{5} = \frac{18}{20} \quad \text{or} \quad \frac{3}{4} \div \frac{5}{6} = \frac{3}{4} \times \frac{6}{5}$$

Activity

Can you see a pattern in the answers to the following fractions?

$$\frac{10}{200} \quad ; \quad \frac{10}{20} \quad ; \quad \frac{10}{2} \quad ; \quad \frac{10}{\frac{1}{2}} \quad ; \quad \frac{10}{\frac{1}{20}}$$

Answer

Simplifying gives

$$\begin{aligned} \frac{10}{200} &= \frac{1}{20} \\ \frac{10}{20} &= \frac{1}{2} \\ \frac{10}{2} &= 5 \\ \frac{10}{\frac{1}{2}} &= 10 \times \frac{2}{1} = 20 \\ \frac{10}{\frac{1}{20}} &= 10 \times \frac{20}{1} = 200 \end{aligned}$$

from which it follows that

$$200 > 20 > 5 > \frac{1}{2} > \frac{1}{20}.$$

The smaller the denominator gets, the larger the answer becomes. Think about it – if you split 10 into groups of two you get only five groups, but if you split 10 into groups of a half each, you get two groups for each unit, therefore for 10 units you get 20 groups.

Since we are looking at division, what about dividing by zero? Look at the following pattern:

$$\frac{1}{1} = 1; \quad \frac{1}{\frac{1}{10}} = 10; \quad \frac{1}{\frac{1}{100}} = 100; \quad \frac{1}{\frac{1}{1\,000}} = 1\,000; \quad \frac{1}{\frac{1}{10\,000}} = 10\,000.$$

We see that 1 is larger than $\frac{1}{10}$, which is **larger** than $\frac{1}{100}$, and so on.

The **smaller** the number by which you divide, the **larger** the answer.

The closer the number by which you divide is to zero, the larger the answer. But you can never divide by zero, because that answer has not been mathematically defined. Let's look at it differently.

Question: How many twos do I have to add together to get six? We know that

$$2 + 2 + 2 = 6.$$

The answer is three twos. Mathematically it can be written as

$$\frac{6}{2} = 3.$$

Question: How many zeros do I have to add together to get six?
Answer: This cannot be determined.

Written mathematically

$$\frac{6}{0}$$

has no meaning.

Activity

Calculate the following:

1. $\frac{12}{\frac{1}{3}}$
2. $\frac{5 \times 4}{\frac{1}{5}}$

Answer

Simplifying gives the following:

1. $12 \times \frac{3}{1} = 36$
2. $5 \times 4 \times 5 = 100$

1.3.2 Addition and subtraction of fractions

How do we calculate

$$\frac{4}{13} + \frac{2}{5}?$$

I heard you! You just said "with a calculator!" Yes, that is the easy way, but do you really understand what you are doing? Before using mechanical aids like calculators, you must make sense of what you are doing. Let us start with something easy and try to find a method, for example

$$\frac{1}{2} + \frac{1}{2} = 1.$$

What we actually did was to write a common denominator for the two denominators. This is of course 2.

Then we fill in the numerator part:

$$\frac{1+1}{2} = \frac{2}{2} = 1.$$

Similarly for

$$\frac{3}{4} + \frac{1}{4} + \frac{6}{4} = \frac{3+1+6}{4} = \frac{10}{4} = \frac{5}{2} = 2\frac{1}{2}.$$

If we add fractions and they all have the same denominator, it is quite easy to do the sum. What about

$$\frac{4}{13} + \frac{2}{5}?$$

A solution is to find the same denominator for both. The easiest way is to multiply the two denominators:

$$13 \times 5 = 65.$$

But $\frac{4}{13}$ is **not** $\frac{4}{65}$! If we multiply the 13 by 5 we must also multiply the 4 by 5 before we add:

$$\frac{4}{13} = \frac{4 \times 5}{13 \times 5} = \frac{20}{65} \quad \text{and} \quad \frac{2}{5} = \frac{2 \times 13}{5 \times 13} = \frac{26}{65},$$

and

$$\frac{4}{13} + \frac{2}{5} = \frac{20+26}{65} = \frac{46}{65}.$$

Shortcut:

$$\frac{4}{13} + \frac{2}{5} = \frac{4 \times 5 + 2 \times 13}{65} = \frac{20+26}{65} = \frac{46}{65}.$$

Activity

Calculate the answers for the following:

1. $\frac{3}{4} - \frac{1}{2}$
2. $\frac{17}{23} + 2$
3. $\frac{8}{15} + \frac{11}{30}$
4. $3\frac{1}{4} - \frac{9}{10}$

Answer

1. Simplifying gives

$$\begin{aligned}\frac{3}{4}-\frac{1}{2} &= \frac{3-1\times 2}{4}\\ &= \frac{1}{4};\end{aligned}$$

or

$$\begin{aligned}\frac{3}{4}-\frac{2}{4} &= \frac{3-2}{4}\\ &= \frac{1}{4};\end{aligned}$$

or

$$\frac{3}{4}-\frac{1\times 2}{4} = \frac{1}{4}.$$

2. Simplifying gives

$$\begin{aligned}\frac{17}{23}+2 &= \frac{17+2\times 23}{23}\\ &= \frac{17+46}{23}\\ &= \frac{63}{23}\\ &= 2\frac{17}{23}.\end{aligned}$$

> Write the improper fraction $\frac{63}{23}$ as a mixed number:
> $$63 \div 23 = 2{,}74,$$
> then
> $$63 - 2 \times 23 = 17,$$
> and
> $$\frac{63}{23} = 2\frac{17}{23}.$$

3. Simplifying gives

$$\begin{aligned}\frac{8}{15}+\frac{11}{30} &= \frac{8\times 2+11}{30}\\ &= \frac{16+11}{30}\\ &= \frac{27}{30}\\ &= \frac{9}{10}.\end{aligned}$$

> Simplify:
> $$\begin{aligned}\frac{27}{30} &= \frac{27\div 3}{30\div 3}\\ &= \frac{9}{10}\end{aligned}$$

4. Simplifying gives

$$\begin{aligned}3\frac{1}{4}-\frac{9}{10} &= \frac{13}{4}-\frac{9}{10}\\ &= \frac{13\times 10-9\times 4}{40}\\ &= \frac{130-36}{40}\\ &= \frac{94}{40}\\ &= \frac{47}{20}\\ &= 2\frac{7}{20}.\end{aligned}$$

> Simplify:
> $$\begin{aligned}\frac{94}{40} &= \frac{94\div 2}{40\div 2}\\ &= \frac{47}{20}\end{aligned}$$
> Write it as a mixed number:
> $$47 \div 20 = 2{,}35,$$
> then
> $$47 - 2 \times 20 = 7,$$
> and
> $$\frac{47}{20} = 2\frac{7}{20}.$$

1.3.3 A decimal world

According to the new *Collins Concise English dictionary*, a decimal is:

1. also called a decimal fraction. A fraction that has an unwritten denominator of a power of ten. It is indicated by a decimal point to the left of the numerator: $0{,}2 = \frac{2}{10}$
2. any number used in the decimal system
3. (a) relating to or using powers of ten
   (b) of the base ten

The word *decimate* (in the ancient Roman Army) means to kill every tenth man in a mutinous section.

Switch on your calculator and key in $1 \div 2 =$, that is, $\frac{1}{2} =$. The answer is 0,5, that is, $\frac{5}{10}$.

(The comma between the 0 and the 5 is called the decimal comma. In most calculators this is given as a decimal point, that is, 0.5).

Every number that you key in is interpreted according to the definition of a decimal as given above. It makes sense, because if you count you have

$$\begin{array}{cccccccccccc} 1; & 2; & 3; & \ldots & 10; & 11; & 12; & 13; & \ldots & 20; & 21 & \\ & & & & & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \\ & & & & & 10+1 & 10+2 & 10+3 & & 10+10 & 10+10+1 & \text{and so on.} \end{array}$$

Without switching on the calculator, one realises that we live in a number world dominated by "tens". Let us write a few fractions as decimals:

$$\begin{aligned} \frac{1}{10} &= 0{,}1 && \text{(10 has one zero, therefore one number after the decimal comma)} \\ \frac{1}{100} &= 0{,}01 && \text{(100 has two zeros, therefore two numbers after the decimal comma)} \\ \frac{1}{1\,000} &= 0{,}001 && \text{(1 000 has three zeros, therefore three numbers after the decimal comma)} \\ \frac{1}{10\,000} &= 0{,}0001 && \text{(10 000 has four zeros, therefore four numbers after the decimal comma)} \\ \frac{1}{100\,000} &= 0{,}00001 && \text{(100 000 has five zeros, therefore five numbers after the decimal comma)} \end{aligned}$$

So how about $\frac{3}{10}$ as a decimal fraction?

Write down 3, which is actually 3,0 and move the decimal comma one place to the left, that is 0,3.

Similarly:

$$\begin{aligned}\frac{3}{10} &= 0{,}3\\ \frac{3}{100} &= 0{,}03\\ \frac{3}{1\,000} &= 0{,}003\\ \frac{3}{10\,000} &= 0{,}0003\\ \frac{3}{100\,000} &= 0{,}00003\end{aligned}$$

Let us go back to the activity on page 14 on fractions:

$$\frac{3}{4} - \frac{1}{2}$$

Switch on your calculator and key in $\frac{3}{4} - \frac{1}{2}$. The answer is 0,25. Then

$$0{,}25 = \frac{25}{100} = \frac{1}{4}$$

which is what we have calculated.

Activity

Go back to the activity on page 14. Use your calculator to calculate the answers.

Answer

Using the calculator, you get the following answers:

1. 0,25
2. 2,73913
3. 0,90
4. 2,35

At this stage we have to talk about rounding. A decimal fraction like 0,43529 must sometimes be rounded to make sense.

If we want to express 0,43529 as a number with two decimal digits, we consider the first three decimal digits and then **round** it to two decimal digits. The decimal fraction

$$0{,}435$$

rounded to two decimal digits is

$$0{,}44.$$

The rule we applied here is

> If the part of the number which is to be discarded begins with a digit greater than or equal to 5, then add 1 to the last digit retained; if not just drop the unwanted part.

Rounding is very important. If the interest rate is 16,25%, then you may not drop the 0,25%. There is a big difference if you calculate the interest on a loan using a 16,25% interest rate or a 16% interest rate. (This will be discussed in the component on the mathematics of finance.)

However, what does it mean if the price of an article is R1,75394? When we are working with money, we will round off to two decimal digits.

Activity

1. Round the following numbers to two decimal digits:
   (a) 3,462
   (b) 8,998
   (c) 7,9747
   (d) 10,495
   (e) 12,034
2. Round the following numbers to three decimal digits:
   (a) 0,0342
   (b) 10,0004
   (c) 4,57849
   (d) 0,39887
   (e) 9,8746

Answer

1. The numbers, rounded to two decimal digits, are
   (a) 3,46
   (b) 9,00
   (c) 7,97 (only look at the third decimal figure)
   (d) 10,50
   (e) 12,03
2. The numbers, rounded to three decimal digits, are
   (a) 0,034
   (b) 10,000
   (c) 4,578 (only look at the fourth decimal figure)
   (d) 0,399
   (e) 9,875

Exercise 1.3

Calculate the answers to the following expressions:

1. $\frac{8}{\frac{1}{4}}$
2. $\frac{12}{\frac{1}{5}}$
3. $\frac{\frac{8}{13}}{\frac{12}{25}}$
4. $\frac{3\frac{1}{2}}{7}$
5. $\frac{8\frac{1}{4}}{\frac{1}{2}}$
6. $\frac{17}{20} + \frac{1}{4} - \frac{3}{5}$
7. $\frac{2}{3} + 5 - \frac{6}{7}$
8. $5\frac{1}{2} + 3\frac{2}{5} - 6\frac{7}{12}$
9. $\frac{3}{4} \div \left(1\frac{5}{6} - \frac{1}{2}\right) + \frac{3}{5}$

Study unit 1.4 Powers and roots

Learning objectives: *On completion of this study unit you should be able to solve problems containing roots and/or powers.*

Although roots and powers are not as much part of our everyday lives as a percentage, they are an extremely important part of mathematics and we must make friends with them. So, without any further ado, let us start.

1.4.1 Powers

Start with multiplying 10 by itself:

$$\begin{aligned} 10 \times 10 &= 100 \\ 10 \times 10 \times 10 &= 1\,000 \\ 10 \times 10 \times 10 \times 10 &= 10\,000 \end{aligned}$$

It seems a very long way of doing it. Fortunately a shorter notation has been developed:

$$\begin{aligned} 10 \times 10 = 10^2 &= 100 && \text{(10 is multiplied 2 times by itself)} \\ 10 \times 10 \times 10 = 10^3 &= 1\,000 && \text{(10 is multiplied 3 times by itself)} \\ 10 \times 10 \times 10 \times 10 = 10^4 &= 10\,000 && \text{(10 is multiplied 4 times by itself)} \end{aligned}$$

The number of times that 10 is multiplied by itself is the power of 10 involved, for example 10^3 is pronounced as "10 to the power three".

Note that $10 = 10^1$ or 10 to the power one. And what about 10^0 or 10 to the power zero? By definition, any number to the power zero is one. So

$$10^0 = 1, \ \ 0^x = 0 \ \text{ for } \ x \neq 0 \ \text{ and } \ 0^0 = 1.$$

Consider 10^3 again, where 10 is called the **base** and 3 is called the **exponent**.

Activity

Which one is the base and which one is the exponent in the following?

$$6^4 \quad ; \quad 3^2 \quad ; \quad 10^8.$$

Answer

For 6^4 : 6 is the base and 4 is the exponent. It is read as "six to the power four".
For 3^2 : 3 is the base and 2 is the exponent. It is read as "three to the power two".
For 10^8 : 10 is the base and 8 is the exponent. It is read as "ten to the power eight".

Multiplication rule for exponents

The calculation

$$100 \times 100 = 10\,000$$

can also be written as

$$10^2 \times 10^2 = 10^4.$$

If the bases are the same, we can just add the exponents to get the answers. But beware! You cannot do this if the bases are different.

Activity

Using $2^3 \times 3^2$ as an example, decide whether the above statement is correct or not.

Answer

Is
$$2^3 \times 3^2 = 2^{3+2} = 2^5 = 32 \text{ or } 3^{3+2} = 3^5 = 243?$$

We have that
$$2^3 = 2 \times 2 \times 2 = 8$$

and
$$3^2 = 3 \times 3 = 9.$$

Therefore
$$2^3 \times 3^2 = 8 \times 9 = 72.$$

It is clear that you can only

add exponents if the bases are the same.

This can also be illustrated if variables are used instead of actual numbers. For example
$$\begin{aligned} x^5 \times x^4 &= x^{5+4} \\ &= x^9 \end{aligned}$$

and
$$\begin{aligned} x^3y^5 \times x^2y^3 &= x^{3+2}y^{5+3} \\ &= x^5y^8. \end{aligned}$$

We can also use actual numbers and variables in the same calculation:
$$\begin{aligned} 4abc \times 3a^2bc \times 2a^3b^2c &= 4 \times 3 \times 2 \times a^{1+2+3} \times b^{1+1+2} \times c^{1+1+1} \\ &= 24a^6b^4c^3. \end{aligned}$$

Simplifying
$$2\left(x^2\right)^4\left(w^3\right)^3\left(x^7\right)^2 w$$

gives
$$\begin{aligned} & 2\left(x^2\right)\left(x^2\right)\left(x^2\right)\left(x^2\right)\left(w^3\right)\left(w^3\right)\left(w^3\right)\left(x^7\right)\left(x^7\right) w \\ = \; & 2x^{2+2+2+2+7+7}w^{3+3+3+1} \\ = \; & 2x^{22}w^{10}. \end{aligned}$$

Another way to do this calculation, is
$$\begin{aligned} & 2\left(x^2\right)^4\left(w^3\right)^3\left(x^7\right)^2 w \\ = \; & 2x^{2\times 4}w^{3\times 3}x^{7\times 2}w \\ = \; & 2x^8w^9x^{14}w \\ = \; & 2x^{8+14}w^{9+1} \\ = \; & 2x^{22}w^{10}. \end{aligned}$$

Activity

Calculate the value of the following:

1. $2a^{\frac{3}{4}} \times 3a^{\frac{1}{4}}$
2. $5p^2q^4\left(p^3\right)^6\left(q^2\right)^4$

Answer

Simplifying gives the following:

1.

$$\begin{aligned} 2a^{\frac{3}{4}} \times 3a^{\frac{1}{4}} &= 2 \times 3 \times a^{\frac{3}{4}+\frac{1}{4}} \\ &= 6a^{\frac{4}{4}} \\ &= 6a \end{aligned}$$

2.

$$\begin{aligned} 5p^2q^4\left(p^3\right)^6\left(q^2\right)^4 &= 5p^2q^4p^{3\times 6}q^{2\times 4} \\ &= 5p^2q^4p^{18}q^8 \\ &= 5p^{2+18}q^{4+8} \\ &= 5p^{20}q^{12} \end{aligned}$$

What is the meaning of a negative power?

If $10^2 = 100$, what is 10^{-2}?

To deal with this we introduce the concept of the **inverse** or **reciprocal** of a number. The inverse or reciprocal of a number is the result obtained when 1 is divided by that number.

The inverse of 10 is $\frac{1}{10}$.

If

$$10 = 10^1$$

then

$$\frac{1}{10} = \frac{1}{10^1} = \frac{1}{10^1} \times \frac{10^{-1}}{10^{-1}} = \frac{10^{-1}}{10^0} = \frac{10^{-1}}{1} = 10^{-1}.$$

The superscript $^{-1}$ is used to indicate the inverse of a number.

Therefore

$$\frac{1}{100} = \frac{1}{10^2} = \frac{1}{10^2} \times \frac{10^{-2}}{10^{-2}} = \frac{10^{-2}}{10^2 \times 10^{-2}} = \frac{10^{-2}}{10^0} = \frac{10^{-2}}{1} = 10^{-2}.$$

Activity

Derive a rule to find the inverse of any number written as a base number with an exponent.

Answer

If

$$10 = 10^{+1}$$

then

$$\frac{1}{10} = 10^{-1}.$$

If

$$100 = 10^{+2}$$

then

$$\frac{1}{100} = 10^{-2}.$$

Then for any number written as a base with an exponent, the inverse is found by using

the *same base*, but *changing* the *sign* of the *exponent*.

Let's look at a few examples:

$$\begin{aligned} \frac{7^{-2}}{10^{-3}} &= \frac{10^3}{7^2} \\ &= \frac{1\,000}{49} \end{aligned}$$

$$\begin{aligned} \frac{10ab^4}{5a^2b^2} &= 2ab^4a^{-2}b^{-2} \\ &= 2a^{1-2}b^{4-2} \\ &= 2a^{-1}b^2 \\ &= \frac{2b^2}{a} \end{aligned}$$

or

$$\begin{aligned} \frac{10ab^4}{5a^2b^2} &= \frac{10abbbb}{5aabb} \\ &= \frac{2b^2}{a} \end{aligned}$$

Activity

Calculate the value of the following:

1. $2\left(2^{-3}\right)$
2. $\dfrac{4^{3x} \times 2^{x+2}}{8^{2x}}$

Answer

Simplifying gives the following:

1.

$$\begin{aligned} 2\left(2^{-3}\right) &= 2\left(\frac{1}{2^3}\right) \\ &= 2\left(\frac{1}{8}\right) \\ &= \frac{1}{4} \end{aligned}$$

or

$$\begin{aligned} 2\left(2^{-3}\right) &= 2^1 \times 2^{-3} \\ &= 2^{1-3} \\ &= 2^{-2} \\ &= \frac{1}{4} \end{aligned}$$

2.

$$\begin{aligned}
\frac{4^{3x} \times 2^{x+2}}{8^{2x}} &= \frac{\left(2^2\right)^{3x} \times 2^x \times 2^2}{\left(2^3\right)^{2x}} \\
&= \frac{2^{6x} \times 2^x \times 2^2}{2^{6x}} \\
&= 2^{6x} \times 2^x \times 2^2 \times 2^{-6x} \\
&= 2^{6x+x-6x} \times 2^2 \\
&= 2^x \times 2^2 \\
&= 4 \times 2^x
\end{aligned}$$

This paragraph concludes with the following rules for exponents:

1. Any number raised to the power of one simply equals that number:

$$2^1 = 2$$

2. Any number raised to the power zero equals one:

$$2^0 = 1$$

3. Numbers with bases and exponents can only be added or subtracted if they have the same base and the same exponent:

$$\begin{aligned}
6y^3 + 2y^3 &= 8y^3, \\
6y^3 - 2y^3 &= 4y^3,
\end{aligned}$$

   but

$$6y^3 + 2y^4 = 6y^3 + 2y^4.$$

4. If the bases are the same, then numbers are multiplied by adding the exponents:

$$2^3 \times 2^4 = 2^7$$

5. If the bases are the same, then numbers are divided by subtracting the exponents:

$$\begin{aligned}
\frac{2^5}{2^3} &= 2^{5-3} \\
&= 2^2
\end{aligned}$$

6. A negative exponent is equivalent to the reciprocal of that number:

$$2^{-5} = \frac{1}{2^5}$$

7. When a number with a base and an exponent is raised to a power, the exponents are multiplied:

$$\begin{aligned}
\left(y^a\right)^b &= y^{ab} \\
\left(x^3y^4\right)^2 &= x^6y^8
\end{aligned}$$

1.4.2 Roots

We know that $10^2 = 10 \times 10 = 100$ and we say "10 to the power 2 is 100". What about going the other way, that is, what number raised to the power 2 is 100? This problem is called determining the **root** of a number.

We reverse the power story by using a root sign

$$\begin{aligned} 10^2 &= 100 \\ \sqrt[2]{100} &= 10. \end{aligned}$$

and 10 is called the "square root" of 100.

Most calculators have a square root key. Convention has it that the little 2 is not written but just the $\sqrt{}$ sign.

Activity

What is the square root of 9?

How would you describe the square root of 9 in words?

Answer

The answer is $\sqrt{9} = 3$.

In words we say, 3 is the number when raised to the power two that will give an answer of 9, that is

$$3^2 = 9.$$

NOTE! The $\sqrt{}$ sign is taken away by squaring the 3. How does this work?

We see that $\sqrt{9}$ is actually $9^{\frac{1}{2}}$ (or 9 to the power half). Then

$$\begin{aligned} \sqrt{9} &= 3 \\ 9^{\frac{1}{2}} &= 3 \\ 9^{\frac{1}{2} \times \frac{2}{1}} &= 3^{\frac{2}{1}} \\ 9 &= 3^2. \end{aligned}$$

What about something like $\sqrt{25 - 16}$ and $\sqrt{16 + 9}$?

Firstly

$$\begin{aligned} \sqrt{25 - 16} &= \sqrt{9} \\ &= 3. \end{aligned}$$

Be careful:

$$\begin{aligned} \sqrt{25} &= 5 \\ \sqrt{16} &= 4 \\ 5 - 4 &= 1, \end{aligned}$$

which is **not** the correct answer as obtained above $(3 \neq 1)$.

Secondly

$$\begin{aligned}\sqrt{16+9} &= \sqrt{25}\\ &= 5.\end{aligned}$$

We also see that

$$\begin{aligned}\sqrt{16} &= 4\\ \sqrt{9} &= 3\\ 4+3 &= 7\end{aligned}$$

which is **also not** the correct answer as obtained above ($5 \neq 7$).

The moral of the story is:

First finish all the calculations inside the square root before taking the square root away by squaring.

At this time, it goes without saying that the **square** root of a number, x, is that number, r, which, when **squared,** becomes x:

$$r^2 = x$$

Consider

$$6^2 = 36.$$

Then

$$\begin{aligned}\sqrt{6\times 6} &= \sqrt{36}\\ &= 36^{\frac{1}{2}}\\ &= 6.\end{aligned}$$

In this case, x is the number, namely 36, and r is the square root, namely 6.

Now for something new!

The **cube root** of a number, x, is a number, r, whose **cube** is x:

$$r^3 = x$$

Consider

$$2^3 = 8.$$

Then

$$\begin{aligned}\sqrt[3]{2\times 2\times 2} &= \sqrt[3]{8}\\ &= 8^{\frac{1}{3}}\\ &= 2.\end{aligned}$$

In this case, x is the number, namely 8, and r is the cube root, namely 2.

The ***n*th root** of a number, x, is a number, r that, when raised to the **power** n is x:

$$r^n = x$$

Consider the case where $n = 5$:

$$2^5 = 32.$$

Then

$$\begin{aligned}\sqrt[5]{2\times 2\times 2\times 2\times 2} &= \sqrt[5]{32}\\ &= 32^{\frac{1}{5}}\\ &= 2.\end{aligned}$$

In this case, x is the number, namely 32, and r is the fifth root, namely 2.

Activity

Determine the following and describe each answer in words:

1. $\sqrt{16}$
2. $\sqrt{25}$
3. $\sqrt[4]{16}$
4. $\sqrt[3]{125}$

Answer

Simplifying gives the following:

1. $\sqrt{16} = 4$
   that is, 4 raised to the power two is 16.
2. $\sqrt{25} = 5$
   that is, 5 raised to the power two is 25.
3. $\sqrt[4]{16} = 2$
   that is, 2 raised to the power four is 16 ($2^4 = 2 \times 2 \times 2 \times 2 = 16$).
4. $\sqrt[3]{125} = 5$
   that is, 5 raised to the power three is 125 ($5^3 = 5 \times 5 \times 5 = 125$).

Exercise 1.4

1. Determine the value of the following:
   (a) 2^3
   (b) 4^2
   (c) 3^{-1}
   (d) 5^{-3}
   (e) $3^2 \times 3^0$
   (f) $2^3 \times 2^2$
   (g) $3^2 \times 4^2$
   (h) $5^{-1} \times 5^2$
2. Calculate:
   (a) $\sqrt{196} + \sqrt{144}$
   (b) $\left(\sqrt{5}\right)^2$
   (c) $\left(\sqrt{64}\right)^2$
   (d) $\left(\sqrt{1}\right)^2$

3. Simplify:
   (a) $\sqrt{100 \div 4}$
   (b) $\sqrt{\dfrac{36}{4}}$
   (c) $2 \times \sqrt{144}$
   (d) $25 - \sqrt{4}$
4. Calculate:
   (a) $\sqrt{18}$
   (b) $\dfrac{\sqrt{45}}{3^{-2}}$
   (c) $\sqrt{\dfrac{9 - x^2}{x^2}}$

Study unit 1.5 Ratios, proportions and percentages

Learning objectives: *On completion of this study unit you should be able to relate numbers to each other using ratios, proportions and percentages.*

In our day-to-day lives we are constantly confronted with situations in which we have to compare numbers: "One out of every three South Africans prefers sugar-free soft drinks", "Twenty percent of our national budget is spent on education and welfare", and so on.

1.5.1 Ratios

A ratio is a way of comparing two (or more) numbers. For example, we say "one out of three" and write 1 to 3 or 1:3 or 1/3. Similarly, we speak of the ratio 2 to 1 or 2:1 or 2/1 in the case of "doubling".

Activity

Write the ratios of each of the following pairs of numbers in each of the three ways:

7 and 16; 22 and 11; 9 and 12.

Answer

Each ratio, written in each of the three ways, is the following:

7 to 16	or	7 : 16	or	7/16
22 to 11	or	22 : 11	or	22/11
9 to 12	or	9 : 12	or	9/12

Note that the last form the ratio is nothing more than a fraction or rational number. Furthermore it is often possible to reduce this ratio to a simpler form, as is the case in the last two examples above, namely

$$\frac{22}{11} = \frac{2 \times 11}{11} = \frac{2}{1}$$

and

$$\frac{9}{12} = \frac{3 \times 3}{3 \times 4} = \frac{3}{4}.$$

Thus, the ratio 22 to 11 is the same as 2 to 1, and 9 to 12 is the same as 3 to 4.

Another way in which a ratio is reduced is by expressing the fraction in decimal form, in which case it is compared to one. To be specific, if we express the three cases above in this way we get the following:

$$\begin{aligned} \frac{7}{16} &= \frac{0{,}4375}{1} \quad \text{or} \quad 0{,}4375 : 1 \\ \frac{2}{1} &= \frac{2}{1} \quad \text{or} \quad 2 : 1 \\ \frac{3}{4} &= \frac{0{,}75}{1} \quad \text{or} \quad 0{,}75 : 1 \end{aligned}$$

Generally speaking, this "comparison to one" formula is the one best suited to calculators and computers.

A room is 4 m wide and 6 m long. What is the ratio of the width to the length, and the ratio of the length to the width? Also express both ratios in the "comparison to one" form, working to three significant figures.

The ratio of the width to the length is 4 to 6 or

$$\frac{4}{6} = \frac{2}{3} \text{ that is, } 0{,}667 : 1.$$

The ratio of the length to the width is 6 to 4 or

$$\frac{6}{4} = \frac{3}{2} \text{ that is, } 1{,}50 : 1.$$

This example illustrates that we must be very careful to place the quantity that we are comparing to, second. In other words, the ratio of width to length is not the same as that of length to width.

Activity

1. Last year the total sales for Water Walker Sailboards was R240 000 while the gross profit was R30 000. Determine the ratio of gross profit to total sales, reduce it, and express it as a comparison-to-one ratio.
2. Percy's Programming Persons, a placement bureau for data processing personnel, has 3 female and 21 male job seekers on its books. What is the ratio of women to men? Reduce this ratio and express it as a comparison-to-one ratio.

Answer

1. The ratio of gross profit to total sales is 30 000 to 240 000 or

$$\frac{30\,000}{240\,000} = \frac{1}{8}$$

   that is, 1 : 8 or 0,125 : 1.
2. The ratio of women to men is

$$3 \text{ to } 21 \text{ or } \frac{3}{21} = \frac{1}{7}$$

   that is, 1 : 7 or 0,143 : 1 (to three decimals).

It is often necessary to divide a number into two parts in order to satisfy a given ratio, as the next example illustrates.

Benjamin and Ernest start a transport service, BETS. Benjamin invests R25 000 and Ernest R20 000, and they agree to divide any profits in the same ratio as their capital investments. At the end of the first month the business shows a net profit of R9 000. What is each partner's share of the profit?

The ratio of their investments is 25 to 20. Now we must divide the profit into two parts in order to satisfy this ratio. This is done as follows:

Add 25 and 20 together to obtain 45 parts in total.

Benjamin's share is R5 000:

$$\frac{25}{45} \times 9\,000 = 5\,000$$

Earnest's share is R4 000:

$$\frac{20}{45} \times 9\,000 = 4\,000$$

Why specifically "45 parts" in total? Would we get the same results if we worked with a reduced ratio? Yes, we would, but let's check:

$$\frac{25}{20} = \frac{5}{4} = \frac{1{,}25}{1}.$$

Consider $\frac{5}{4}$ first. Add 5 and 4 together to obtain 9 parts in total.

For Benjamin

$$\frac{5}{9} \times 9\,000 = 5\,000$$

and for Ernest

$$\frac{4}{9} \times 9\,000 = 4\,000$$

as before.

Or, working with 1,25 to 1, we add 1,25 and 1 together to obtain 2,25 parts in total.

Benjamin's and Ernest's shares are respectively

$$\frac{1{,}25}{2{,}25} \times 9\,000 = 5\,000 \quad \text{and} \quad \frac{1}{2{,}25} \times 9\,000 = 4\,000.$$

In other words, it does **not matter** whether we work with the **given ratio**, or a **reduced ratio**, or a **comparison-to-one ratio**. We may choose whichever is most convenient.

Activity

Cleansweep, an office cleaning company, lands a big new contract and has to expand its staff from a total of 240 cleaners and 10 supervisors to a total of 400. If the same ratio of supervisors to cleaners is to be maintained, how many supervisors are required?

Answer

The ratio of supervisors to cleaners is

$$10 \text{ to } 240, \text{ or } 1 \text{ to } 24$$

in reduced form (or 0,04167 to 1).

Thus we add 1 and 24 together to obtain 25 parts in total.

The number of supervisors required, is

$$\frac{1}{25} \times 400 = 16.$$

The number of cleaners is

$$\frac{24}{25} \times 400 = 384.$$

Note that we could have worked with the original ratio:

$$10 + 240 = 250$$

parts in total or the comparison-to-one ratio and obtained the same results.

To check we calculate the ratio of supervisor to cleaners. This is

$$16 \text{ to } 384$$

which is

$$0{,}04167 \text{ to } 1.$$

Before moving on to the concept of proportion, I must point out that ratios are not just used to compare two numbers. Three or more numbers can be compared using ratios, as the next example shows.

Benjamin and Ernest's Transport Service is doing well, so that when they recognise a need for a courier service for sensitive documents they decide to establish a new firm – Discreet Deliveries. They go into partnership with Salina. Their initial investments are as follows: Benjamin R7 500, Ernest R10 000 and Salina R2 500. If they agree to divide profits in the same ratio as their capital investments, how will they share out the first month's profit of R8 000?

The ratio of their investments is

$$7\,500 \quad \text{to} \quad 10\,000 \quad \text{to} \quad 2\,500$$

or, in reduced form,

$$3 \quad \text{to} \quad 4 \quad \text{to} \quad 1$$

which we can also write as

$$3 : 4 : 1.$$

We add 3, 4 and 1 together to obtain 8 parts in total.

Thus, Benjamin's share is R3 000:

$$\frac{3}{8} \times 8\,000 = 3\,000,$$

Ernest's share is R4 000:

$$\frac{4}{8} \times 8\,000 = 4\,000$$

and Salina's share is R1 000:

$$\frac{1}{8} \times 8\,000 = 1\,000.$$

The next activity requires the same approach.

Activity

Much to the chagrin of all his relatives, Uncle Wilfred's will stipulates that R240 000 of his estate should be divided between his three pets, Percy the parrot, Bozo the bulldog and Sullivan the Siamese cat, in the ratio 7 : 5 : 4. How much did each pet receive?

Answer

Add 7, 5 and 4 together to obtain 16 parts in total.

Thus, Percy receives R105 000:

$$\frac{7}{16} \times 240\,000 = 105\,000,$$

Bozo receives R75 000:

$$\frac{5}{16} \times 240\,000 = 75\,000$$

and Sullivan receives R60 000:

$$\frac{4}{16} \times 240\,000 = 60\,000.$$

To check, note that

$$105 : 75 : 60$$

reduces to

$$1{,}75 : 1{,}25 : 1$$

(after division by 60), which is the comparison to one form of

$$7 : 5 : 4.$$

1.5.2 Proportion

In ordinary language the words **ratio** and **proportion** are often used as synonyms and, indeed, for most purposes this slight confusion is acceptable and of little consequence. Mathematically speaking, however, there is a subtle difference.

A **proportion** is a statement of equality between ratios. Typically, statements of proportionality arise when ratios are reduced or when they are converted from one frame of units to another. For example

$$\frac{15}{25} = \frac{3}{5} \quad \text{or} \quad 15 : 25 = 3 : 5$$

which is read 15 is to 25 as 3 is to 5. Also

$$24 : 16 : 8 = 3 : 2 : 1$$

which is read 24 is to 16 is to 8 as 3 is to 2 is to 1.

Statements of proportionality are used all the time to scale up, or down, the values for a known or accepted situation to new values for new situations.

If you are paid R1 580 for 5 working days, how much would you earn for 17?

You will probably do this calculation without thinking, but I would like you to take a few minutes, as the basic step is the same for all proportion-type problems.

The ratio of rand to days is

$$1\,580 \quad \text{to} \quad 5$$

which reduces to

$$316 : 1$$

(rand per day or rand to days).

This is, of course, nothing more than the rate of pay. To obtain the wage for 17 days simply multiply this rate by 17 to obtain

$$316 \times 17 = 5\,372.$$

The wage for 17 days is thus R5 372. Finally, check your answer by making sure that the two ratios are equal, that is,

$$\frac{5\,372}{17} = \frac{1\,580}{5}.$$

Activity

Property tax is often assessed on a proportional basis. Suppose that a certain municipality charges R350 per year for every R10 000 assessed valuation. What would the annual tax be on a property valued at R650 000?

Answer

Again, we first determine the basic rate, which is, in fact, the comparison to one ratio. The ratio of tax to valuation is

$$350 \quad \text{to} \quad 10\,000$$

or

$$0{,}035 : 1.$$

Thus, the annual tax is calculated as

$$0{,}035 \times 650\,000 = 22\,750.$$

The annual tax is R22 750.

Check:

$$\frac{22\,750}{650\,000} = \frac{350}{10\,000}.$$

You will have noticed that the basic trick is to determine the comparison-to-one ratio, which we call the rate. "How do I know to which number I must compare? In other words, which number comes second in the ratio?" you may wonder. The answer is simple – **the second number is always the one related to the multiplicand in the final step**, for example days in the first example and valuation in the second example. And remember to always check your answer as shown above. With these hints as a guide try the next two activities.

Activity

1. You undertake a trip in your new car and find that for the first 300 km you use 25 litres of petrol. How many litres do you anticipate you will need for the next 360 km?
2. You obtain the licence for Gobbling Goblin, the latest computer game, and make a profit of R52 000 on the first 650 games sold. Assuming all prices remain the same, how much do you expect to make on the anticipated sales of 3 000 games, for the next quarter?

Answer

1. The ratio required is
$$25 \text{ to } 300 \text{ or } 0{,}08333 \text{ to } 1$$
(l per km). You would therefore expect to use
$$0{,}08333 \times 360 = 30{,}0$$
litres for the next 360 km.
Check:
$$\frac{30}{360} = \frac{25}{300}$$
2. The ratio required is
$$52\,000 \text{ to } 650 \text{ or } 80 \text{ to } 1$$
(rand profit per game sold). Your anticipated profit for the next quarter is therefore
$$80 \times 3\,000 = 240\,000.$$
The anticipated profit is R240 000.
Check:
$$\frac{240\,000}{3\,000} = \frac{52\,000}{650}$$

1.5.3 Percent

There is probably not a day that goes by during which you will not hear some reference to the term *percent*. It is certainly the most commonly used way of indicating the relative size of two numbers and you are probably quite familiar with basic calculations using percentages. The recommended calculator has a percentage key. Please see Tutorial Letter 101 on how to use it.

The term *percent* comes from the Latin words *per* and *centum* and means "by the hundred", or fractional part of one hundred or the ratio of a number to one hundred. It is denoted by the symbol %.

For example:

$$\begin{array}{rcl} 25\% & \text{means} & \dfrac{25}{100} \\[2ex] 67\% & \text{means} & \dfrac{67}{100} \\[2ex] 33\frac{1}{3}\% & \text{means} & \dfrac{33\frac{1}{3}}{100} \\[2ex] 150\% & \text{means} & \dfrac{150}{100} \end{array}$$

You decide to buy a one-bedroom flat and apply to the bank for a loan. The purchase price is R380 000. The building society requires you to put down a deposit of 25%, and it is prepared to grant you a bond for the remaining 75%. How much must you deposit?

You will recognise that this is a proportion-type calculation which we dealt with in the last paragraph. A 25% deposit means that for every R100 of the purchase price you must deposit R25. We first calculate the ratio 25 : 100 which is 0,25 : 1. Your total deposit is calculated as

$$0{,}25 \times 380\,000 = 95\,000.$$

Your deposit is R95 000.

As this example illustrates, in order to use percent in computations we need to express it as a decimal (or a fraction). Fortunately, in the case of percent this is much simpler than for ratios in general. You will recall that division by 100 simply implies shifting the decimal point two places to the left. For example:

$$\begin{array}{rcccl} 6\% & = & \dfrac{6}{100} & = & 0{,}06 \\[2ex] 10{,}5\% & = & \dfrac{10{,}5}{100} & = & 0{,}105 \\[2ex] 122{,}7\% & = & \dfrac{122{,}7}{100} & = & 1{,}227 \end{array}$$

Of course, if you are not sure in any specific instance you can always consult your calculator – or you can always enter a percent as a number followed by ÷ 100, if you want to be on the safe side.

Activity

1. During a sale at Wendy's Wardrobe, goods are marked down 30%. How much will a dress that normally costs R188 now be?
2. A salesman receives a commission that is $2\frac{1}{2}\%$ of his sales. If his sales for the month total R320 000, what is his commission?

Answer

1. The percentage 30% is 0,30, so the amount off is

$$188 \times 0{,}30 = 56{,}40.$$

   The amount off is R56,40. The price to be paid is

$$188 - 56{,}40 = 131{,}60.$$

   The dress will now cost R131,60.

2. The percentage $2\frac{1}{2}\%$ is 0,025. Thus, the commission is calculated as

$$320\,000 \times 0{,}025 = 8\,000.$$

   His commission is R8 000.

Sometimes, (as the next example illustrates), it is necessary to calculate the percentage rate applicable to a problem.

Frederick had R1 960 income tax deducted from his gross monthly pay. If his gross pay was R8 800, what percentage of this is his income tax? Express your answer to one decimal place.

We reduce the rates

$$1\,960 : 8\,800 \quad \text{to} \quad 0{,}2227 : 1.$$

However, percent means parts per hundred, so multiply by 100 to obtain 22,27%. Thus 22,3% was paid as income tax.

Activity

1. A survey reveals that in a small town, with 11 275 families, 3 712 families own television sets. What percentage of families owns sets?
2. Sky High, a construction company, last year realised a net profit of R552 500 on a contract for R8 500 000. What percentage of the contract value was the net profit?

Answer

1. The ratio that owns TV sets is

$$3\,712 : 11\,275$$

   which reduces to

$$0{,}329 : 1.$$

   Multiplying by hundred gives 32,9%.

2. The ratio of net profit to contract value is

$$552\,500 : 8\,500\,000$$

   which reduces to

$$0{,}065 : 1.$$

   Multiplying by 100 gives 6,5%.

So far we have dealt with two types of calculation that involve percent. In the first case we were given a base number and a percentage and asked to calculate the corresponding amount, for example 10% of "so much" is ... . In the second case we were asked to determine what percentage one amount is of another. Another form of calculation involving percent is when we know both the percentage and the final number but not the initial, or **base** number, and want to work back to it. The following example illustrates this.

A video recorder is offered as a special for R3 600 and it is stated that this is 75% of the usual price. What was the price before discount?

The percentage 75% is 0,75. Divide 3 600 by 0,75, which gives 4 800, which is the normal price. This is because

$$\begin{aligned} 75\% \text{ of the price} &= 3\,600 \\ 0{,}75 \times \text{price} &= 3\,600 \\ \text{price} &= \frac{3\,600}{0{,}75} \\ &= 4\,800. \end{aligned}$$

The normal price is R4 800. To check, calculate the ratio of

$$3\,600 : 4\,800$$

which reduces to

$$0{,}75 : 1.$$

Multiplying by 100 gives 75%.

Activity

1. Patrick's Paintshop advertises "all goods at 80% of normal price". For a quantity of paint and brushes you are charged R422,40. What would the normal price have been?
2. Benjamin and Ernest calculate that 55% of last year's total expenses was for salaries. If the salary account was R825 000, what were the total expenses?

Answer

1. The percentage 80% is 0,8. Dividing 422,40 by 0,8 gives 528, which would have been the normal price.

   Check that 422,40 : 528 reduces to 80%:

$$\begin{aligned} 80\% \text{ of the price} &= 422{,}40 \\ 0{,}80 \times \text{price} &= 422{,}40 \\ \text{price} &= \frac{422{,}40}{0{,}80} \\ &= 528{,}00 \end{aligned}$$

   The normal price is R528,00.

2. The percentage 55% is 0,55. Dividing 825 000 by 0,55 gives 1 500 000 for the total expenses. Check that 825 000 : 1 500 000 reduces to 55%:

$$\begin{aligned} 55\% \text{ of expenses} &= 825\,000 \\ 0{,}55 \times \text{expenses} &= 825\,000 \\ \text{expenses} &= \frac{825\,000}{0{,}55} \\ &= 1\,500\,000 \end{aligned}$$

The total expenses are R1 500 000.

Study unit 1.6 Signs, notations and counting rules

Learning objectives: *On completion of this study unit you should know and be able to use*

- *all the basic signs and notations*
- *the counting rules*

1.6.1 Signs and notation

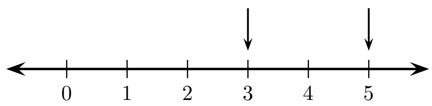
The signs in mathematics, without which we cannot live, are given below:

1. **= (equal)**

   This sign is used in equations and indicates that the left-hand side is the same as the right-hand side. An equation is sometimes like $3 \times 2 = 6$.

2. **< (less than)**

   Three is less than five, is written as $3 < 5$ (3 is to the left-hand side of 5 on the number line). The sign < indicates "smaller than".

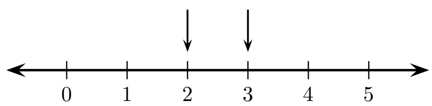
   

3. **≤ (less than or equal)**

   This means that the left-hand side can be less than the right-hand side or equal to it. Then $a \leq 5$ means that a can be any value smaller than 5, but that it can also be equal to 5.

4. **> (greater than)**

   Three is greater than two, is written as $3 > 2$ (3 is to the right-hand side of 2 on the number line).

5. **≥ (greater than or equal)**

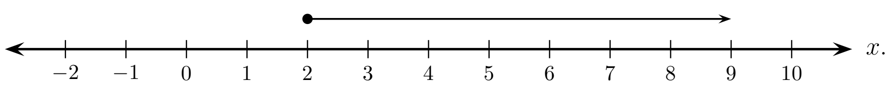
   This means that the left-hand side of an expression can be more than the right-hand side of the expression, but can also be equal to it.

   Then $a \geq 5$ means that a can be any value greater than 5, but that it can also be equal to 5.

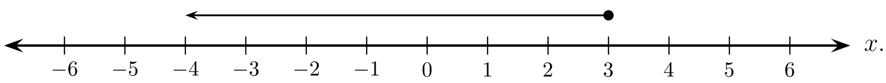
6. **Play on words**

   Often the signs are not given, only words like "... at least ...", "... no more than ..." and "at the most ...". How can we write this using signs only?

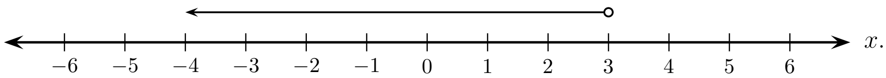
   At least 2: this means **two or more** or **two included** and we write it as $x \geq 2$ and show it graphically as



No more than 3: this means **everything less than 3 and 3 included** and we write it as $x \leq 3$ and show it graphically as



Less than 3: this means that 3 is excluded and we write it as $x < 3$ and show it graphically as



Note that the words "**between** 2 and 5" are often confusing. It must be stated clearly whether the endpoints, 2 and 5, are included.

7. Up to now we have worked with positive integers only to illustrate the symbols and signs. What about something like $1 - 2$? We use our picture with the steps inside the building.

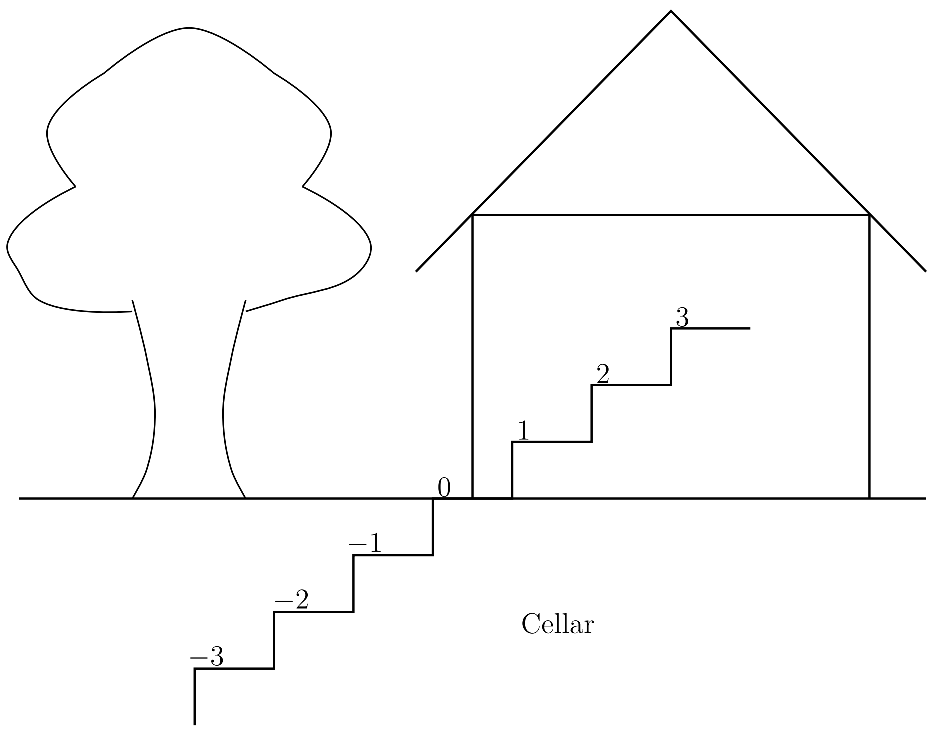


Figure 1.6.1

Stand on step 1 and go **down** two steps (the minus indicating the direction in which you must go) and you will end up on step -1.

Thus, $1 - 2 = -1$.

What about $3 - -1$?

Start from step $+3$. The $-$ indicates you have to go down. Go down to step -1.

The total number of steps that you have descended is 4. Hence the **distance** is 4 and $3 - -1 = 4$.

There must be some pattern or rule for this. One can't go on climbing up and down steps forever!

Let's take another look at $1 - 2$ and $3 - -1$.

$1 - 2$ is the same as $1 - +2$ (look at natural numbers again).

The $-$ changes the $+$ next to it so that we have a $-$.

Next: $3 - -1$.

This $-$ changes the next $-$ to a $+$.

What if it were $3 + 1$? We see that $3 + 1$ is the same as $3 + + 1 = 4$.

> The minus *changes* the sign next to it.
> A plus does *not change* the sign next to it:
> that sign *remains the same* and the plus *disappears.*

What happens to pluses and minuses in multiplication and division?

Consider 3×-2. That is, $+3 \times -2$. Now arrange all the signs in front of the figures. This gives

$$\begin{aligned} + - 3 \times 2 &= -3 \times 2 \\ &= -6 \text{ (here the plus disappears but not the minus).} \end{aligned}$$

Likewise

$$\begin{aligned} -2 \times -2 &= - - 2 \times 2 \\ &= +4 \text{ (the minus changes the sign next to it).} \end{aligned}$$

Let us look at what happens with $-2 \times -2 \times -2$ and $-2 \times -2 \times -2 \times -2$:

Firstly:

$$\begin{aligned} & -2 \times -2 \times -2 \\ = & - - -2 \times 2 \times 2 \\ = & -8. \end{aligned}$$

Consider only the $-$ signs:

First: $- - -$

Then: $+ -$

Then: $-$

Secondly:

$$\begin{aligned} & -2 \times -2 \times -2 \times -2 \\ = & - - - - 2 \times 2 \times 2 \times 2 \\ = & +16. \end{aligned}$$

Again consider only the $-$ signs:

First: $- - - -$

Then: $+ - -$ or $- - +$

Then: $- -$ then $+ +$

Then: $+$ then $+$

Do you see what I see?

- When there is an **even** number of minuses in multiplication (ie 2, 4, 6, etc), the sign becomes a plus (+) and the answer is a **positive** number.
- When there is an **uneven** number of minuses (ie 1, 3, 5, 7, etc) the sign becomes a minus (−) and the answer is a **negative** number.

Summation

The summation symbol, $\sum$, is an important symbol used when working with numbers. This is a very useful sign, because it is the mathematical shorthand sign for adding. The Greek capital letter for S is $\sum$ and is pronounced as sigma. Suppose we want the sum of all the integers between 1 and 10, 1 and 10 included. Instead of saying: "Add all the integers from 1 to 10 and write down the sum", we write:

$$\sum_{i=1}^{10} i$$

Below the $\sum$ sign we have $i = 1$. This indicates that you must add from the first position.

On top we have 10. This is the last value that i can be. Thus

$$\sum_{i=1}^{10} i = 1 + 2 + 3 + \ldots + 10.$$

If we have a set of observations we write

$$\sum_{i=1}^{n} x_i$$

where x_i indicates an observation. The subscript $i = 1$ indicates that we start counting from the first observation and add all the observations up to the n-th observation. Thus

$$\sum_{i=1}^{10} x_i = x_1 + x_2 + \ldots + x_{10}.$$

If we have a set of observations that must be added, we just write $\sum x_i$. When no confusion is possible, it is not necessary to add the subscripts and superscripts.

Activity

If $x_1 = 3$, $x_2 = 1$ and $x_3 = 5$, calculate the following:

1. $\displaystyle\sum_{i=1}^{5} 1$
2. $\displaystyle\sum_{i=1}^{3} x_i$

Answer

1. The answer is

$$\begin{aligned} \sum_{i=1}^{5} 1 &= 1 + 1 + 1 + 1 + 1 \\ &= 5. \end{aligned}$$

2. The answer is

$$\begin{aligned} \sum_{i=1}^{3} x_i &= x_1 + x_2 + x_3 \\ &= 3 + 1 + 5 \\ &= 9. \end{aligned}$$

1.6.2 Counting rules

1.6.2.1 The multiplication rule

The registration numbers of cars in Gauteng consist of two letters, two numeric figures (numbers), two letters and GP (for Gauteng Province). An example is BC 38 YM GP (with Gauteng's coat of arms in front of the GP).

How many registrations are possible if no vowels (ie a, e, i, o and u) may be used? Remember that a letter and a number may be used more than once.

There are 21 letters that may be used if no vowel is allowed.

For the first two-letter part:
The first position can be filled in 21 different ways.
The second position can be filled in 21 different ways.

For the numeric part we may use any of the 10 figures, 0, 1, 2, ..., 9. The first position can be filled in 10 ways and the second position can be filled in 10 ways.

For the last two-letter part:
The first position can be filled in 21 different ways.
The second position can be filled in 21 different ways.

The total number of registrations possible is

$$21 \times 21 \times 10 \times 10 \times 21 \times 21 = 19\,448\,100,$$

which is quite a lot!

The multiplication rule

If an operation can be performed in n_1 ways, and thereafter it is performed in any one of these ways, a second operation can be performed in n_2 ways, and after this second operation has been performed in any one of these ways a third operation can be performed in n_3 ways, and so on for k operations, then the k operations can be performed in

$$n_1 \times n_2 \times n_3 \times \ldots \times n_k$$

ways.

Activity

If a parking garage has five entrances and three exits, in how many ways can a motorist enter and leave the garage?

Answer

The motorist can enter in 5 different ways.
The motorist can leave in 3 different ways.

The total number of ways in which he can enter and leave is

$$5 \times 3 = 15.$$

1.6.2.2 Permutations

The exclamation mark, !, has a special task in mathematics.

Suppose we want to arrange or order five numbers. The first number can be placed in any of five places. This can be done in five ways. Then four places are left and the second number can take any of the four places. Similarly the third number can take any of three places. The total number of ways is

$$5 \times 4 \times 3 \times 2 \times 1 = 120.$$

A shorthand way of writing this is 5! and we say: "5 factorial".

> The *factorial* of a non-negative integer n, denoted by $n!$, is the product of all positive integers less than or equal to n.

Therefore:

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$$

Note:

$$0! = 1$$

Then 5! can also be written as

$$\begin{array}{l} 5 \times 4! \text{ or} \\ 5 \times 4 \times 3! \text{ or} \\ 5 \times 4 \times 3 \times 2! \text{ or} \\ 5 \times 4 \times 3 \times 2 \times 1! \end{array}$$

(*See Tutorial Letter 101 on how to use the ! key on the recommended calculator.*)

Activity

Seven horses run in a race. What is the total number of ways in which they can complete the race?

Answer

The first horse can take any of seven places.
The next horse can take any of six places.

The total number of ways is

$$7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5\,040.$$

How do we handle the following situation?

A list of 10 investment possibilities are presented to the directors of a company. Each director must order the five projects he considers as the best in order of importance. How many different arrangements are possible?

It is clear that order of placement is of importance and that only five must be chosen out of the ten possibilities. When order is of importance, we use permutations.

We want to determine the number of permutations of five out of ten objects.

The formula is

$$\begin{aligned} {}_{10}P_5 &= \frac{10!}{(10-5)!} \\ &= \frac{10!}{5!} \\ &= \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5!}{5!} \\ &= 30\,240. \end{aligned}$$

In general, the number of permutations of x objects out of m objects is

$${}_mP_x = \frac{m!}{(m-x)!}.$$

The notations ${}^{10}P_5$ or $P(10,5)$ are also used.

(*See Tutorial Letter 101 on how to calculate* ${}_{10}P_5$, *using the recommended calculator.*)

Activity

How many arrangements are possible for the first three places in a race with eight horses?

Answer

The answer is

$$\begin{aligned} {}_8P_3 &= \frac{8!}{(8-3)!} \\ &= \frac{8!}{5!} \\ &= \frac{8 \times 7 \times 6 \times 5!}{5!} \\ &= 8 \times 7 \times 6 \\ &= 336. \end{aligned}$$

1.6.2.3 Combinations

When order of placement is not important, we use combinations instead of permutations.

Suppose we have four workers of equal competence. In how many different ways can we select two workers?

Suppose the workers are A, B, C and D.

The possible choices of two out of four is

A B	B C	C D
A C	B D	
A D		

The formula used here is

$$\begin{aligned} {}_4C_2 &= \frac{4!}{2!\,2!} \\ &= \frac{4\times 3\times 2!}{2!\times 2!} \\ &= \frac{4\times 3}{2!} \\ &= \frac{12}{2} \\ &= 6. \end{aligned}$$

Generally, the combination of x objects out of m possible objects is

$$ {}_mC_x = \frac{m!}{(m-x)!\,x!}$$

or ${}^mC_x$ or $C(m,x)$ or $\binom{m}{x}$.

If order of placement were important, then we would have:

A B	B A	B C	C B	C D	D C
A C	C A	B D	D B		
A D	D A				

(*See Tutorial Letter 101 on how to calculate ${}_4C_2$, using the recommended calculator.*)

Activity

In how many ways can a police captain choose any three of his seven detectives for a special assignment?

Answer

Order of placement is not important. The possible number of combinations is

$$\begin{aligned} {}_7C_3 &= \frac{7!}{(7-3)!\,3!} \\ &= \frac{7!}{4!\,3!} \\ &= \frac{5\,040}{24\times 6} \\ &= 35. \end{aligned}$$

1.6.2.4 The difference between permutations and combinations

We sometimes get confused between "permutation" and "combination" – which one is which?

Here is an easy way to remember: **permutation sounds complicated**, doesn't it? And it is. With permutations, every little detail matters. Alice, Bob and Charlie is different from Charlie, Bob and Alice.

Combinations, on the other hand, are pretty easy going. The details don't matter. Alice, Bob and Charlie is the same as Charlie, Bob and Alice.

Permutations are for **lists** (order matters) and **combinations** are for **groups** (order doesn't matter).

Permutations

Let's start with permutations, or **all the possible ways** of doing something. We are going to care about every last detail, including the order of items. Let's say we have eight people:

1: Alice (A = 1)
2: Bob
3: Charlie
4: David
5: Eve
6: Frank
7: George
8: Horatio

How many ways can we pick a gold, silver and bronze medal for "Best friend in the world"?

Let's look at the following representation:

(A)	(B)	(C)	(D)	(E)	(F)	(G)	(H)	8 choices
	(B)	(C)	(D)	(E)	(F)	(G)	(H)	7 choices
		(C)	(D)	(E)	(F)	(G)	(H)	6 choices

We're going to use permutations since the order in which we hand out these medals matters. Here is how it breaks down:

- Gold medal: 8 choices: A B C D E F G H. Let's say A wins the gold.
- Silver medal: 7 choices: B C D E F G H. Let's say B wins the silver.
- Bronze medal: 6 choices: C D E F G H. Let's say C wins the bronze.

We had eight choices at first, then seven, then six. The total number of options was

$$8 \times 7 \times 6 = 336.$$

Let's look at the details. We had to order three people out of eight. To do this, we started with all options (eight) then took them away one at a time (seven, then six) until we ran out of medals.

We know the factorial is

$$8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1.$$

Unfortunately, that does too much! We only want $8 \times 7 \times 6$. How can we "stop" the factorial at 5?

This is where permutations get cool: notice how we want to get rid of $5 \times 4 \times 3 \times 2 \times 1$. What's another name for this? Five factorial!

So, if we do $\frac{8!}{5!}$ we get

$$\begin{aligned} \frac{8!}{5!} &= \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1} \\ &= 8 \times 7 \times 6. \end{aligned}$$

And why did we use the number 5? Because it was left over after we picked three medals from eight. So, a better way to write this would be

$$\frac{8!}{(8-3)!}$$

where $\frac{8!}{(8-3)!}$ is just a fancy way of saying: "Use the first three numbers of eight!". If we have $\boldsymbol{m}$ items in total and want to pick $\boldsymbol{x}$ of them in a certain order, we get

$$\frac{m!}{(m-x)!}.$$

This just means: "Use the first x numbers of m!", which is the permutation formula. Suppose you have $\boldsymbol{m}$ items and want to find the number of ways $\boldsymbol{x}$ items can be ordered, then

$$_mP_x = \frac{m!}{(m-x)!}.$$

Combinations

Combinations are easy. Order doesn't matter. You can mix it up and it looks the same. Let's say I am a cheapskate and cannot afford separate gold, silver and bronze medals. In fact, I can only afford empty tin cans.

How many ways can I give three tin cans to eight people?

Well, in this case, the order we pick people does not matter. If I give a can to Alice, Bob and then Charlie, it is the same as giving one to Charlie, Alice and then Bob. Either way, they're going to be equally disappointed.

This raises an interesting point – we have got some redundancies here. Alice Bob Charlie is equal to Charlie Bob Alice. For a moment, let's just figure out how many ways we can rearrange three people.

Well, we have three choices for the first person, two for the second, and only one for the last. So we have $3 \times 2 \times 1$ ways to rearrange three people.

Wait a minute ... this is looking a bit like a permutation!

It is indeed! If you have m people and you want to know how many arrangements there are for **all** of them, you simply work out m factorial or $m!$

So, if we have three tin cans to give away, there are 3! or 6 **variations** for every choice of three picked out of eight. If we want to figure out how many combinations we have, we just **create all the permutations and divide by the number of variations for each permutation.** In our case, we get 336 permutations (from above) and we divide by the 6 **variations** for each permutation and get

$$336 \div 6 = 56.$$

The general formula is

$$_mC_x = \frac{_mP_x}{x!}$$

which means: "Find all the ways to pick x people from m, and divide by the $x!$ variations." Writing this out, we get our **combination formula**, or the number of ways to combine x items from a set of m:

$$\begin{aligned} _mC_x &= \frac{_mP_x}{x!} \\ &= \frac{m!}{(m-x)!} \div x! \\ &= \frac{m!}{(m-x)!} \times \frac{1}{x!} \\ &= \frac{m!}{(m-x)!\,x!} \end{aligned}$$

A few examples

Here are a few examples of combinations (order does not matter) and permutations (order matters).

1. *Combination:* Picking a team of three people from a group of ten gives

$$\begin{aligned} _{10}C_3 &= \frac{10!}{(10-3)!\,3!} \\ &= \frac{10!}{7!\,3!} \\ &= 120. \end{aligned}$$

   Permutation: Picking a president, a vice-president and water boy from a group of ten gives

$$\begin{aligned} _{10}P_3 &= \frac{10!}{(10-3)!} \\ &= \frac{10!}{7!} \\ &= 720. \end{aligned}$$

2. *Combination:* Choosing three desserts from a menu of ten.
   Permutation: Listing your three favourite desserts, in order, from a menu of ten.

Exercise 1.5

1. Use the symbols $<$, $>$ or $=$ to make the following true:
   (a) -5 ___ -2
   (b) 9 ___ -2
   (c) -100 ___ 7
   (d) -6 ___ -12
   (e) 2 ___ 0
   (f) $+3$ ___ 3
2. Write down the following and complete the missing parts:
   (a) $x < 5$ and $x \geq 0$ can also be written as 0 ___ x ___ 5.
   (b) $x \geq -3$ and $x < 3$ can also be written as -3 ___ x ___ 3.
   (c) $-6 < x \leq 5$ can also be written as ___ and ___.
   (d) $0 \leq x < 6$ can also be written as ___ and ___.
3. Graph the following inequalities on a number line (x is an integer):
   (a) $x \geq -3$
   (b) $x < 5$
   (c) $-3 < x \leq 7$
   (d) $x \geq -4$ and $x < 5$
4. Solve the following:
   (a) $\dfrac{7!}{5!}$
   (b) $(14 - 11)! + 2! \times 4!$
5. If $x_1 = 3$, $x_2 = 5$, $x_3 = 4$ and $x_4 = 2$, solve the following:
   (a) $\sum_{i=1}^{4} x_i$
   (b) $\sum_{i=2}^{3} x_i$
   (c) $\sum_{i=1}^{4} x_i^2$
6. (a) How many four-letter words (including those not making sense) are possible if a character may appear more than once in the same word?
   (b) How many meals are possible if there is a choice of four starters, ten main courses and six desserts?
   (c) Any three people out of twelve can be chosen for a committee. How many possible arrangements are there?
   (d) How many four-letter **words** are possible if a letter may not occur more than once in the same word (including words not making sense)?

Study unit 1.7 Units and measures

Learning objectives: *On completion of this study unit you should*

- *convert units of length/distance (eg mm to m)*
- *convert units of area (eg mm^2 to m^2)*
- *convert units of volume (eg mm^3 to m^3)*

It is difficult to imagine a world without a system for measuring things. You would not know the distance to another town, the capacity in litres of a fuel tank; whether your body mass is within limits and so on.

The earliest measuring systems originated in the barter system and units corresponded with things like the length or size of a hand or foot. Many countries had their own measuring systems. In the modern world, which is characterised by international trade and extended industrial and technological development, it became necessary to have a common system. Consequently, a modern international system, the *Système Internationale d'Unités,* also known as SI, was developed. South Africa was one of the first countries to accept this system.

The following table will give us a good indication of the SI sytem:

Number	**Power**	**Common name**	**SI name**	**SI abbreviation**
10	10^1	ten	deca -	D
100	10^2	hundred	hecta -	h
1 000	10^3	thousand	kilo -	k
1 000 000	10^6	million	mega -	M
1 000 000 000	10^9	milliard*	giga -	G
1 000 000 000 000	10^{12}	billion	tera -	T
0,1	10^{-1}	tenth	deci -	d
0,01	10^{-2}	hundredth	centi -	c
0,001	10^{-3}	thousandth	milli -	m
0,000001	10^{-6}	millionth	micro -	μ
0,000000001	10^{-9}	milliardth	nano -	n
0,000000000001	10^{-12}	billionth	pico -	p

*(American word for milliard is billion.)

1.7.1 Length

The standard unit for length is the metre. The SI abbreviation for metre is m. Therefore we write 10 metres as 10 m.

Lengths in the SI system:

10 millimetres	(mm)	=	1 centimetre	(cm)
10 centimetres	(cm)	=	1 decimetre	(dm)
10 decimetres	(dm)	=	1 metre	(m)
10 metres	(m)	=	1 decametre	(dam)
10 decametres	(dam)	=	1 hectometre	(hm)
10 hectometres	(hm)	=	1 kilometre	(km)

We normally use km, m, cm and mm. Then:

$$1\,\text{km} = 1\,000\,\text{m}$$

$$1\,\text{m} = 100\,\text{cm} = 1\,000\,\text{mm}$$

In the textile industry where, among other things, clothes are manufactured, centimetres are used for technical reasons. As a result, all body lengths are measured in centimetres.

Distance is another word for length and it is convention that distances between places are measured in kilometres.

Activity

Convert the following:

1. 24 cm to mm
2. 416 m to km
3. 20 km to m
4. 8 214 mm to m
5. 12,4 m to cm
6. 1 932,3 m to km
7. A km to cm
8. 50 m + 0,5 km to m

Answer

1. We know that

$$1\,\text{cm} = 10\,\text{mm}.$$

Thus, to convert 24 cm to mm you multiply by 10:

$$24 \times 10 = 240$$

24 cm is therefore 240 mm.

2. We know that

$$1\,000\,\text{m} = 1\,\text{km}.$$

Divide both sides by 1 000 to get

$$\frac{1\,000}{1\,000} = \frac{1}{1\,000}.$$

Then,

$$1\,\text{m} = \frac{1}{1\,000}\,\text{km}.$$

Thus, to convert 416 m to km you divide by 1 000:

$$\frac{416}{1\,000} = 0{,}416$$

416 m is therefore 0,416 km.

3. We know that

$$1\,\text{km} = 1\,000\,\text{m}.$$

Thus, to convert 20 km to m you multiply by 1 000:

$$20 \times 1\,000 = 20\,000$$

20 km is therefore 20 000 m.

4. We know that

$$1\,000\,\text{mm} = 100\,\text{cm} = 1\,\text{m},$$

thus

$$1\,000\,\text{mm} = 1\,\text{m}.$$

Divide both sides by 1 000 to get

$$\frac{1\,000}{1\,000} = \frac{1}{1\,000}.$$

Then,

$$1\,\text{mm} = \frac{1}{1\,000}\,\text{m}.$$

Thus, to convert 8 214 mm to m you divide by 1 000:

$$\frac{8\,214}{1\,000} = 8{,}214$$

8 214 mm is therefore 8,214 m.

5. We know that

$$1\,\text{m} = 100\,\text{cm}.$$

Thus, to convert 12,4 m to cm you multiply by 100:

$$12{,}4 \times 100 = 1\,240$$

12,4 m is therefore 1 240 cm.

6. We know that

$$1\,000\,\text{m} = 1\,\text{km}.$$

Divide both sides by 1 000 to get

$$\frac{1\,000}{1\,000} = \frac{1}{1\,000}.$$

Then,

$$1\,\text{m} = \frac{1}{1\,000}\,\text{km}.$$

Thus, to convert 1 932,3 m to km you divide by 1 000:

$$\frac{1\,932{,}3}{1\,000} = 1{,}9323$$

1 932,3 m is therefore 1,93 km.

7. We know that

$$1\,\text{km} = 1\,000\,\text{m}$$

and

$$1\,\text{m} = 100\,\text{cm}$$

thus

$$1\,000 \times 100 = 100\,000.$$

Then

$$1\,\text{km} = 100\,000\,\text{cm}.$$

Thus, to convert A km to cm you multiply by 100 000:

$$A \times 100\,000 = 100\,000A$$

A km is therefore $100\,000A$ cm.

8. In number 3 we have seen that to convert km to m you multiply by 1 000. For 0,5 km:

$$0{,}5 \times 1\,000 = 500$$

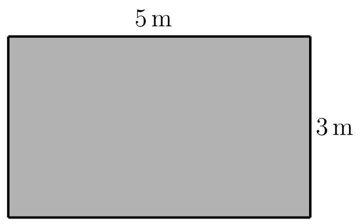
0,5 km is therefore 500 m.
Then

$$50 + 500 = 550.$$

50 m plus 0,5 km is therefore 550 m.

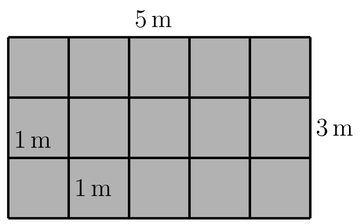
1.7.2 Area

Area is another word for the size of a surface. How do you specify the size of a surface? Suppose we have a metal plate which is five metres long and three metres wide.



The SI unit most often used for area is square metre. One square metre is the size of a square where each side of the square is one metre in length.

The area of the plate is the number of squares that can fit onto the plate. If each side of a square is 1 m in length, how many squares can fit on the plate?



The number of squares is $5 \times 3 = 15$. (Count them!)

The area is then 15 square metres. Thus, to calculate the area of a rectangle, we multiply the length by the width and the answer is given as a "square . . . ". The unit following the "square" will depend on the unit we are working with.

Convention has it that square metres are written as m^2. How would you write square kilometres? Easy – km^2!

The area of a square tile with sides of one metre each is calculated as

$$\begin{aligned} area &= length \times width \\ &= 1 \times 1 \end{aligned} \qquad \boxed{\text{m} \times \text{m} = \text{m}^2}$$

or $1\,\text{m}^2$ (one **square metre**).

If $1\,\text{m} = 100\,\text{cm}$, then the area can also be calculated as

$$\begin{aligned} area &= 100 \times 100 \qquad \boxed{\text{cm} \times \text{cm}} \\ &= 10\,000. \qquad \boxed{\text{cm}^2} \end{aligned}$$

Thus, $1\,\text{m}^2$ is equal to $10\,000\,\text{cm}^2$.

When we are talking about land and its size, then something like km^2 or m^2 can be difficult to visualise. For this purpose a hectare is used. A hectare is the area of a square where each side has a length of $100\,\text{m}$. The area of a hectare is thus

$$\begin{aligned} area &= 100 \times 100 \qquad \boxed{\text{m} \times \text{m}} \\ &= 10\,000. \qquad \boxed{\text{m}^2} \end{aligned}$$

Thus one hectare equals $10\,000\,\text{m}^2$.

Activity

How many hectares are in one square kilometre?

Answer

The area of a square piece of land where each side is one kilometre long is calculated as

$$\begin{aligned} area &= length \times width \\ &= 1 \times 1 \end{aligned} \qquad \boxed{\text{km} \times \text{km} = \text{km}^2}$$

or $1\,\text{km}^2$ (one **square kilometre**).

If $1\,\text{km} = 1\,000\,\text{m}$, then the area of the land can also be written as

$$\begin{aligned} area &= 1\,000 \times 1\,000 \qquad \boxed{\text{m} \times \text{m}} \\ &= 1\,000\,000. \qquad \boxed{\text{m}^2} \end{aligned}$$

Thus, $1\,\text{km}^2$ is equal to $1\,000\,000\,\text{m}^2$.

We also know that a hectare is $10\,000\,\text{m}^2$, thus

$$\frac{1\,000\,000}{10\,000} = 100. \qquad \boxed{\frac{\text{m}^2}{\text{m}^2}}$$

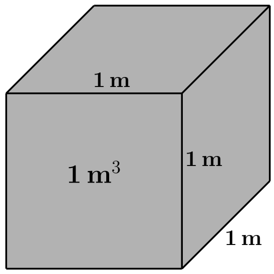
There are 100 hectares in **one square kilometre** ($1\,\text{km}^2$).

A last note:

Instead of saying every time the width is x and the length is y, we talk about a rectangle of size x by y and write $x \times y$.

1.7.3 Volume

The SI unit for volume is the cubic metre (m^3). The volume of a cube where each side is one metre in length, is one cubic metre.



The volume of the cube with sides on one metre each is calculated as

$$volume = length \times width \times height$$
$$= 1 \times 1 \times 1 \qquad \boxed{\text{m} \times \text{m} \times \text{m} = \text{m}^3}$$

or $1\,\text{m}^3$ (**cubic metre**).

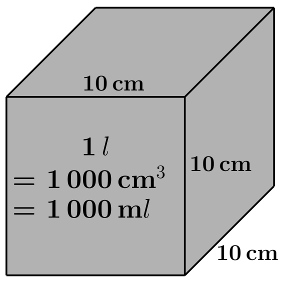
We know from paragraph 1.7.1 that 1 m is equal to 100 cm. Then the volume of the cube can also be written as

$$volume = 100 \times 100 \times 100 \qquad \boxed{\text{cm} \times \text{cm} \times \text{cm}}$$
$$= 1\,000\,000 \qquad \boxed{\text{cm}^3}$$
$$= 10^6.$$

Thus, $1\,\text{m}^3$ is equal to $1\,000\,000\,\text{cm}^3$. Therefore:

to change from m^3 to cm^3 we must multiply by 10^6
to change from cm^3 to m^3 we must divide by 10^6

Another generally used unit of volume is the litre. Say you have a plastic cubic container with sides (length, width and height) of 10 cm each. Then you can pour in one litre of fluid into that container:



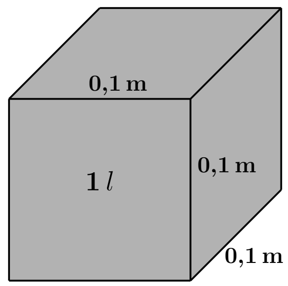
The volume of $1\,l$ is calculated as

$$volume = length \times width \times height$$
$$= 10 \times 10 \times 10 \qquad \boxed{\text{cm} \times \text{cm} \times \text{cm}}$$
$$= 1\,000 \qquad \boxed{\text{cm}^3}$$
$$= 10^3.$$

Thus, $1\,l$ equals $10^3\,\text{cm}^3$ or $1\,000\,\text{cm}^3$. One liter can also be written as $1\,000\,\text{m}l$ ($1\,\text{cm}^3 = 1\,\text{m}l$). Therefore:

to change from litres to cm^3 (or ml) we must multiply by 10^3
to change from cm^3 (or ml) to litres we must divide by 10^3

Consider the previous figure where the lengths of sides of the cube were 10 cm each. From paragraph 1.7.1 we know that 10 cm is equal to 0,1 m. Thus, the cube can also be represented as



The volume of 1 l is calculated as

$$\begin{aligned} volume &= length \times width \times height \\ &= 0{,}1 \times 0{,}1 \times 0{,}1 && \text{m} \times \text{m} \times \text{m} \\ &= 0{,}001. && \text{m}^3 \end{aligned}$$

Thus, 1 l equals 0,001 m^3. Therefore:

to change from litres to m^3 we must divide by 1 000 or 10^3
to change from m^3 to litres we must multiply by 1 000 or 10^3

Activity

Convert the following:

1. 2 386 cm^3 to l
2. 283 l to m^3
3. 2 m^3 to cm^3
4. 146 cm^3 to m^3
5. 2,4 l to cm^3
6. 1 800 cm^3 to l

Answer

1. To convert 2 386 cm^3 to litres we divide by 10^3 or 1 000:

$$\frac{2\,386}{1\,000} = 2{,}386$$

2 386 cm^3 is thus equal to 2,386 l.

2. To convert 283 l to m^3 we divide by 10^3 or 1 000:

$$\frac{283}{1\,000} = 0{,}283$$

283 l is thus equal to 0,283 m^3.

3. To convert $2\,\text{m}^3$ to cm^3 we multiply by 10^6:

$$2 \times 10^6 = 2\,000\,000$$

$2\,\text{m}^3$ is thus equal to $2 \times 10^6\,\text{cm}^3$.

4. To convert $146\,\text{cm}^3$ to m^3 we divide by 10^6:

$$\frac{146}{10^6} = 146 \times 10^{-6}$$

$146\,\text{cm}^3$ is thus equal to $146 \times 10^{-6}\,\text{m}^3$ or $1{,}46 \times 10^{-4}\,\text{m}^3$.

5. To convert $2{,}4\,l$ to cm^3 we multiply by 10^3:

$$2{,}4 \times 10^3 = 2\,400$$

$2{,}4\,l$ is thus equal to $2{,}4 \times 10^3\,\text{cm}^3$ or $2\,400\,\text{cm}^3$.

6. To convert $1\,800\,\text{cm}^3$ to litres we divide by 10^3 or $1\,000$:

$$\frac{1\,800}{1\,000} = 1{,}8$$

$1\,800\,\text{cm}^3$ is thus equal to $1{,}8\,l$.

Distance, volume and area – a summary:

$1\,\text{km}$	$=$	$1\,000\,\text{m}$ ($10^3\,\text{m}$)		
$1\,\text{m}$	$=$	$1\,000\,\text{mm}$		
$1\,\text{m}^2$	$=$	$10^4\,\text{cm}^2$	$=$	$10^6\,\text{mm}^2$
$1\,\text{m}^3$	$=$	$10^6\,\text{cm}^3$	$=$	$10^9\,\text{mm}^3$
$1\,\text{cm}^2$	$=$	$100\,\text{mm}^2$	$=$	$10^2\,\text{mm}^2$
$1\,\text{m}^3$	$=$	$10^3\,l$	$=$	$1\,000\,l$
$1\,l$	$=$	$10^3\,\text{cm}^3$		
$1\,\text{m}l$	$=$	$1\,\text{cm}^3$		

Exercise 1.6

1. Determine the area of the following rectangles:
   (a) $25\,\text{mm} \times 24\,\text{mm}$
   (b) $1{,}2\,\text{km} \times 375\,\text{m}$
   (c) $4{,}4\,\text{m} \times 450\,\text{mm}$
   (d) $225\,\text{mm} \times 122\,\text{mm}$
2. Complete the following:
   (a) $1\,\text{km}^2$ equals ____ m^2.
   (b) $1\,\text{m}^2$ equals ____ mm^2.
   (c) $1\,\text{cm}^2$ equals ____ mm^2.
   (d) 1 ha equals ____ m^2.
   (e) $1\,\text{m}^2$ equals ____ cm^2.
   (f) $24{,}6\,\text{cm}^2$ equals ____ m^2.
   (g) $24\,869{,}3\,\text{mm}^2$ equals ____ m^2.
3. The length of a rectangle is 9 m and the area is $45\,\text{m}^2$. Determine the width of the rectangle.
4. A rectangular container has a length of 1 m, a width of 1 m and a depth of 1 m. Calculate in cubic metres the volume of the container. Calculate the number of litres that can be poured into the container.
5. The length, width and depth of a fuel tank are 60 cm, 50 cm and 20 cm respectively. The price of fuel is R9,16 per litre. How much will it cost to fill the tank?

COMPONENT 2

Collection, presentation and description of data

After completion of this component you should be able to explain how to collect data, how to present them visually and how to calculate simple measurements of data.

CONTENTS

Introduction: Problem sensing

It is very important to solve problems quickly and effectively. However, you cannot solve a problem if you don't know that it exists! A good manager will detect a problem and solve it before it becomes a crisis. But how does he manage to do that? William Pounds, a management specialist, did some research on this in 1969. He found that the following four strategies were used to detect the existence of a possible problem.

1. **Historical information**

   If we assume that the best prediction of the future is the immediate past, then we can expect continuity of performance – whatever happened in the recent past, should continue in the near future. If there are differences, for example, the safety record of this quarter is worse than that of the last quarter, or this month's telephone bill is much higher than last month's bill, one should pay immediate attention to these differences.

2. **Budgetary information**

   Strategic plans and budgets provide detailed information of what is expected. If the actual situation does not correspond to these, there may be hidden problems.

3. **External sources**

   In the late 1970s, early 1980s, potential customers advised General Motors in America to produce reliable, fuel-efficient, front-wheel-drive cars, or otherwise run the risk of losing a large share of the market. When customers are dissatisfied with products or a service, it is often an indication of a problem within the organisation.

4. **Extra-organisational strategy**

   Trade journals, conferences, or a manager's informal network of contacts within the industry often identify problems. "How do we compare with published performance levels?" "Should we consider the new procedures demonstrated at the trade show?" "Why do the annual reports of our competitors look so much better than our own?"

Once a problem has been identified, a problem-solving procedure can be initiated.

Managers use data to understand their organisation and identify problems. However, the data collected must be organised and summarised to make them usable.

In this component we discuss a few simple, but effective, methods to make sense out of available information.

Study unit 2.1 Data collection

Learning objectives: *On completion of this study unit you should know and be able to explain*

- *what a sample and a population are*
- *what simple random sampling, stratified random sampling and systematic sampling are*

Radial, a tyre company, advertises that its XXX tyres, generally known as "Triple X", will complete at least 65 000 kilometres before one of the four tyres will no longer meet the minimum safety requirements. However, several complaints have been received that the tyres completed 50 000 kilometres only when the minimum requirements were not met.

Radial sells directly to the public and it is company policy to keep a record of customers. During the past two years that it has been manufacturing the XXX tyres, it has sold 2 600 sets. Radial feels that it just does not have the time, personnel or money to locate and question all 2 600 of it's customers. It feels that if it could question 100 customers this should give it a good idea of what the actual situation is.

In other words, Radial will draw a **sample** of 100 out of the **population** of 2 600. "Sample" and "population" are words that are inevitable when we wish to obtain data.

But before I define sample and population, consider the following definition:

A *variable* is any property or characteristic that can be measured or observed.

A variable can take on a range of different values.

For example, the distance completed with a set of tyres is different for each customer and therefore the observations **vary** continually. Therefore, distance completed is a variable.

Activity

Write down three other examples of variables.

The *sample unit* is the item that is measured or counted with respect to the variable being studied.

Radial's sample unit is a set of tyres to be measured for the minimum safety requirements.

Activity

What are the sample units for the variables that you have written down in the above activity?

A *population* is the set of all the elements or items being studied.

In Radial's case the 2 600 sets of XXX tyres sold, are the population.

A *sample* is a representative group or a subset of the population.

The 100 sets of tyres that Radial will investigate will form the sample.

What is very important is that this sample **must** be representative of the population. How does one manage this? There are several methods for selecting a representative group.

2.1.1 Types of sampling

2.1.1.1 Simple random sampling

A good sample requires that each item in the population has an equal and independent chance to be included in the sample.

A simple random sample is a sample that has been chosen in such a way that each possible sample containing the same number of observations has the same chance of being drawn.

One method for drawing a simple random sample is to allocate a number to each item in the population. Then use a computer to generate a sequence of random numbers and use these numbers to identify items in the population to be included in the sample.

Activity

A printing company, Printapage, has 30 clients with the following outstanding balances (in rand).

Account No.	Balance	Account No.	Balance
1	25	16	0
2	0	17	102
3	605	18	215
4	1 010	19	429
5	527	20	197
6	34	21	159
7	245	22	279
8	59	23	115
9	667	24	27
10	403	25	27
11	918	26	291
12	801	27	16
13	227	28	0
14	0	29	402
15	47	30	570

The following random numbers are available:

22; 17; 83; 57; 27; 54; 19; 51; 39; 59; 84 and 20.

Use these numbers to draw a random sample of size 5 out of the 30 customer accounts.

Answer

Since the total number of elements in the population is 30, a number larger than 30 is of no use.

The sample units are the numbers of the accounts to be drawn. These are

22; 17; 27; 19 and 20.

The outstanding debts are

279; 102; 16; 429 and 197.

Activity

A political candidate wishes to determine the opinions of the voters in his ward. He decides on a sample of size 20. Using random numbers he chooses 20 telephone numbers from the telephone directory for a telephonic survey. Is this procedure correct? Give a reason for your answer.

Answer

All residents may not have telephones, and all the numbers of those who do have telephones may not be included in the telephone directory. Therefore, such a sample cannot be considered to be random.

2.1.1.2 Stratified random sampling

Simple random sampling requires no prior (a priori) knowledge of the population and can therefore be done with relatively little effort. It can, however, happen that all the elements drawn for the sample are nearly homogeneous or alike. This may cause the conclusions about the population to be biased.

If, however, you have prior information about the population, you can rule out this problem to some degree and consider more correct information about the population by making use of stratified random sampling.

The population is divided into mutually exclusive sets or **strata**. This means that a specific element may belong to one group or **stratum** only.

The strata must be chosen in such a way that there will be large differences between the strata, but small differences between the elements within the same stratum. Now simple random samples are taken from each stratum. Often the number of elements taken from each stratum is proportional to the size of that stratum.

Activity

Divide Printapage's 30 customers into three strata as follows:

Stratum	**Balance (rand)**
1	< 200
2	$200 - 600$
3	> 600

A proportional sample of size 12 must be drawn from the population. How would you do it?

Answer

The data divided into three strata look as follows:

	Account number	Balance
Stratum 1		
	1	25
	2	0
	6	34
	8	59
	14	0
	15	47
	16	0
	17	102
	20	197
	21	159
	23	115
	24	27
	25	27
	27	16
	28	0
Stratum 2		
	5	527
	7	245
	10	403
	13	227
	18	215
	19	429
	22	279
	26	291
	29	402
	30	570
Stratum 3		
	3	605
	4	1 010
	9	667
	11	918
	12	801

The number of elements (frequency) in each stratum is

Stratum	Frequency
1	15
2	10
3	5
	30

To draw a proportional sample of size 12, the number of items that must be drawn from stratum 1 is

$$\frac{15}{30} \times 12 = 6,$$

the number of items that must be drawn from stratum 2 is

$$\frac{10}{30} \times 12 = 4$$

and the number of items that must be drawn from stratum 3 is

$$\frac{5}{30} \times 12 = 2.$$

2.1.1.3 Systematic sampling

Systematic sampling starts at a randomly chosen starting point in the population. Then each subsequent kth element is chosen.

A political candidate wishes to determine the opinions of the voters in his ward. He has a list of voters available. He could, for example, start with voter number six and thereafter select every tenth voter to complete a questionnaire.

Activity

What is the disadvantage of such a method?

Answer

If the variable being considered is periodic in nature, systematic sampling may lead to misleading results. If, for example, we are to estimate a shop's sales and use a 1-in-7 systematic sampling design, it could happen that only sales figures for Saturdays are selected. Sales would then be overestimated.

Systematic sampling is convenient especially when the size of the population is not known.

There are several more types of sampling techniques but they fall outside the scope of this module.

Study unit 2.2 Presentations

Learning objectives: *On completion of this study unit you should*

- *distinguish between qualitative and quantitative data*
- *draw up a frequency table*
- *draw a histogram*
- *draw a pie chart*
- *draw a cumulative frequency polygon*
- *draw a stem-and-leaf diagram*

Radial is happy that its sample is representative. The sample elements are (in thousands)

61	38	19	58	66	64	72	66	64	75
42	24	77	70	46	69	45	46	59	16
59	72	46	50	37	78	66	75	66	67
98	64	64	72	59	88	75	67	45	61
61	77	29	26	62	80	22	83	53	51
82	16	78	34	70	50	69	54	78	77
45	62	45	58	90	86	62	50	56	58
51	32	86	40	62	70	40	67	80	66
14	54	51	54	67	64	69	51	48	72
32	46	22	30	61	74	74	62	64	75

What do we now do with this? These are just a lot of figures of which one can make neither head nor tail!

Before we try doing something with these figures, let us first consider the different types of data one may get.

There are two main groups of data – qualitative and quantitative. Qualitative data are characterised by categorical answers such as yes or no, male or female, and so on. Quantitative data are characterised by numerical values.

Activity

Is Radial's data quantitative or qualitative?

Answer

They are numerical and therefore quantitative.

Quantitative data can be divided into two groups, discrete data and continuous data. Discrete data include everything that can be considered as a separate unit because of its nature. Examples are numbers sold, number of consumers, number of job opportunities, and such like – that is, everything that you can count on your fingers.

Continuous data are usually the result of a measurement and are not fixed isolated points. There can be a whole range of values between any two values. Length and mass are two perfect examples. Time and temperature measurements are further examples.

Activity

Classify the data collected in each of the following questions:

1. Do you own a TV set? Yes ☐ No ☐
2. How many TV sets do you own?
3. How many kilometres did you drive with your set of Radial tyres?
4. What was your electricity bill last month?

Answer

1. The data is qualitative.
2. This is discrete quantitative data.
3. This is continuous quantitative data.
4. This is continuous quantitative data.

We identified Radial's data as quantitative and continuous.

If we could now only envisage the data, we would be able to form a better idea of what is going on.

The histogram is one of the most common ways to visually represent data. A histogram is a graphical presentation of a frequency table. And a frequency table? That is a table in which the data are grouped into intervals.

The steps needed to draw up a frequency table are given below.

1. **Find the range, *R*, of the data as**

$$R = \text{maximum value} - \text{minimum value}.$$

2. **Decide on the number of intervals.**

   When the number of intervals is too few or too many, one cannot get a good idea of the distribution of the data. It is not always easy to decide on the number of intervals to use. A good number of intervals to use is

$$\frac{R}{10}$$

   if R is large, but any number between five and eight is acceptable. Do not use fewer than five, otherwise you will not get a good idea of what is going on.

3. **Determine the width of the intervals as**

$$\frac{R}{\text{number of intervals}}.$$

   The width must be a whole number – this makes it easier to determine the limits of the intervals.

4. **Determine the interval limits.**

   The limits must be such that there is no doubt in which interval a value falls. For example, when we are working with Radial's data, we cannot choose intervals such as

$$\begin{array}{c} 55 - 65 \\ 65 - 75. \end{array}$$

Why not? Where would you place a value of 65?

For the mathematical manipulations that we will be doing with grouped data, we should not work with intervals such as

55 – just smaller than 65
65 – just smaller than 75.

What does "just smaller than 65" mean?

The rule that we will use is to take this to mean half a unit less. Then there can be no confusion as to which interval a value belongs. The lower limit of the first class must be a value which is smaller than the minimum data value. The upper limit of each interval is set to the lower limit of the succeeding interval.

5. **Tabulate the data.**

Let us now set up a frequency table of Radial's data.

1. The minimum value is 14.
   The maximum value is 98.
   The range is calculated as
   $$98 - 14 = 84,$$
   thus $R = 84$.
2. The number of intervals is
   $$\frac{R}{10} = 8{,}4.$$
   Use 8 intervals.
3. The interval width is
   $$\begin{aligned} \frac{R}{8} &= \frac{84}{8} \\ &= 10{,}5. \end{aligned}$$
   Use 11 as width.
4. Determine the interval limits.

   The minimum value is 14. Half a unit less is 13,5. To obtain the upper limit we add the width to the lower limit: 13,5 + 11 = 24,5.

   Therefore, the first interval is 13,5 – 24,5.

   The second interval also starts with 24,5 and also has a width of 11.

   Therefore, its upper limit is 24,5 + 11 = 35,5.

Thus the intervals are

13,5 – 24,5 } 11
24,5 – 35,5
35,5 – 46,5
46,5 – 57,5
57,5 – 68,5
68,5 – 79,5
79,5 – 90,5 } 11
90,5 – 101,5.

5. The only remaining thing to do is to group the data into the intervals.

   Now go back to page 68 and consider the first four sample elements in the first row, which are 61, 38, 19 and 58. Our aim is to find in which one of the following intervals they belong.

Interval	
13,5 – 24,5	19 ← Fits in the 1st interval because it is greater than 13,5 and less than 24,5.
24,5 – 35,5	
35,5 – 46,5	38 ← Fits in the 3rd interval because it is greater than 35,5 and less than 46,5.
46,5 – 57,5	
57,5 – 68,5	61 and 58 ← Fit in the 5th interval because they are greater than 57,5 and less than 68,5.
68,5 – 79,5	
79,5 – 90,5	
90,5 – 101,5	

   Instead of writing 19, 38, 61 and 58 in their corresponding intervals, we represent them with a line, |, as follows:

Interval	
13,5 – 24,5	|
24,5 – 35,5	
35,5 – 46,5	|
46,5 – 57,5	
57,5 – 68,5	||
68,5 – 79,5	
79,5 – 90,5	
90,5 – 101,5	

   The fifth element in a group of lines is indicated by a line drawn across the group: ~~||||~~ represents a group of five. The frequency of an interval is the total number of elements falling into that interval.

Activity

Set up the complete frequency table for Radial.

Answer

The frequency table is

Interval		**Frequency**
13,5 – 24,5	~~||||~~ ||	7
24,5 – 35,5	~~||||~~ |	6
35,5 – 46,5	~~||||~~ ~~||||~~ |||	13
46,5 – 57,5	~~||||~~ ~~||||~~ |||	13
57,5 – 68,5	~~||||~~ ~~||||~~ ~~||||~~ ~~||||~~ ~~||||~~ ~~||||~~	30
68,5 – 79,5	~~||||~~ ~~||||~~ ~~||||~~ ~~||||~~ ||	22
79,5 – 90,5	~~||||~~ |||	8
90,5 – 101,5	|	1
		100

It is clear that the highest frequency is in the interval 57,5 – 68,5. This shows that most of the customers got between 57 500 and 68 500 kilometres per set of tyres.

Activity

What percentage of the customers got

1. 80 000 kilometres or more per set of tyres?
2. 46 000 kilometres or less per set of tyres?

Answer

1. The distance of 80 000 kilometres or more is represented by the intervals

$$79{,}5 - 90{,}5$$

and

$$90{,}5 - 101{,}5.$$

The sum of the frequencies is

$$8 + 1 = 9.$$

Thus

$$\frac{9}{100} \times 100 = 9\%.$$

This means that 9% of the customers got 80 000 kilometres or more per set of tyres. Notice that

$$\frac{9}{100}$$

is called the relative frequency.

2. The distance of 46 000 kilometres or less is accounted for by the first three intervals. The sum of the frequencies is

$$7 + 6 + 13 = 26.$$

Thus

$$\frac{26}{100} \times 100 = 26\%.$$

This means that 26% of the customers got 46 000 kilometres or less per set of tyres.

Now we can graphically represent the frequency table by drawing the interval lengths on a horizontal axis and the frequencies on a vertical axis.

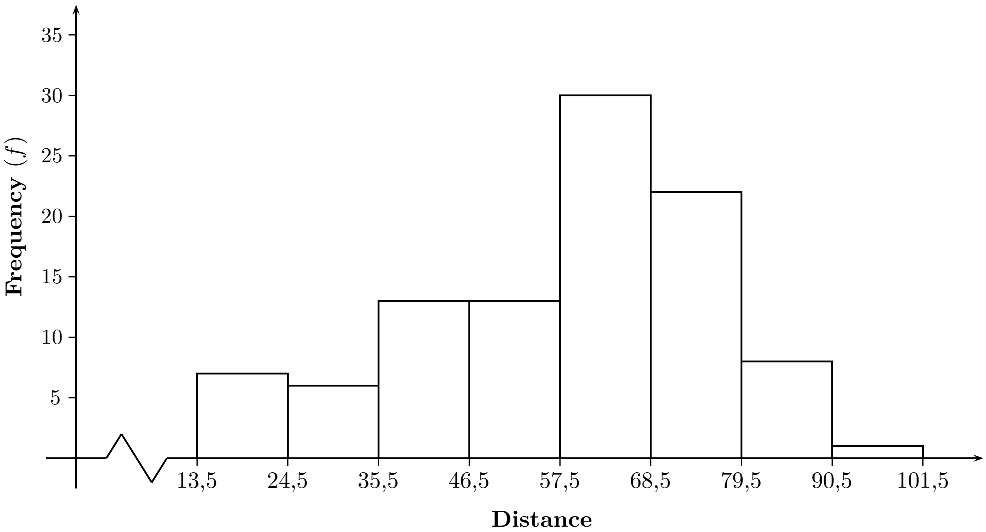


Figure 2.2.1

(Notice that the horizontal axis starts at 0 and that the zigzag line is there to break the line in order to prevent a huge space from appearing to the left of the actual graph.)

Another way of representing data is the pie chart.

A pie chart is drawn as a circle and the "slices" within the circle present the relative frequencies expressed as a percentage. It is often difficult to draw a pie chart by hand. In Radial's case one needs to divide the circle into 100 equal slices – not an easy task!

The pie chart for Radial is more or less as follows.

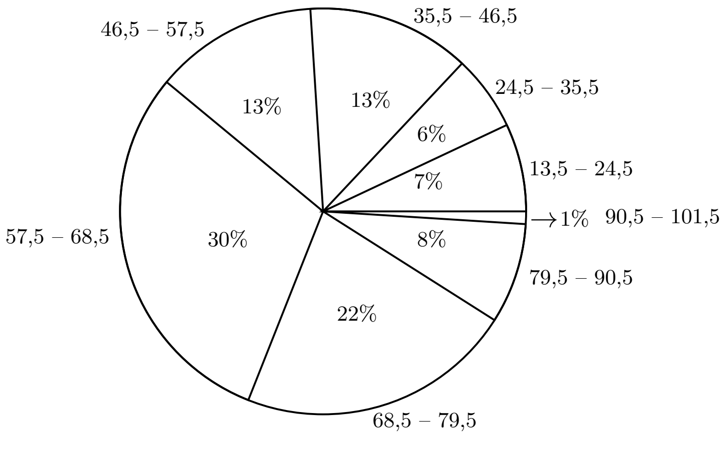


Figure 2.2.2

We have calculated that 26% of the customers drove 46 000 kilometres or less with a set of tyres. Such information can be presented graphically if we first obtain the "cumulative less than" table. Such a table is set up from the frequency table, setting the upper limits to "less than . . . ".

The cumulative frequency table for Radial is as follows:

Upper limit	Cumulative frequency	
$<$ 24,5	7	
$<$ 35,5	13	$(7 + 6 = 13)$
$<$ 46,5	26	$(7 + 6 + 13 = 26)$
$<$ 57,5	39	$(7 + 6 + 13 + 13 = 39)$
$<$ 68,5	69	$(7 + 6 + 13 + 13 + 30 = 69)$
$<$ 79,5	91	
$<$ 90,5	99	
$<$ 101,5	100	

I am sure that you have realised that cumulative means "added up"!

This information can now be represented by a cumulative frequency polygon as follows:

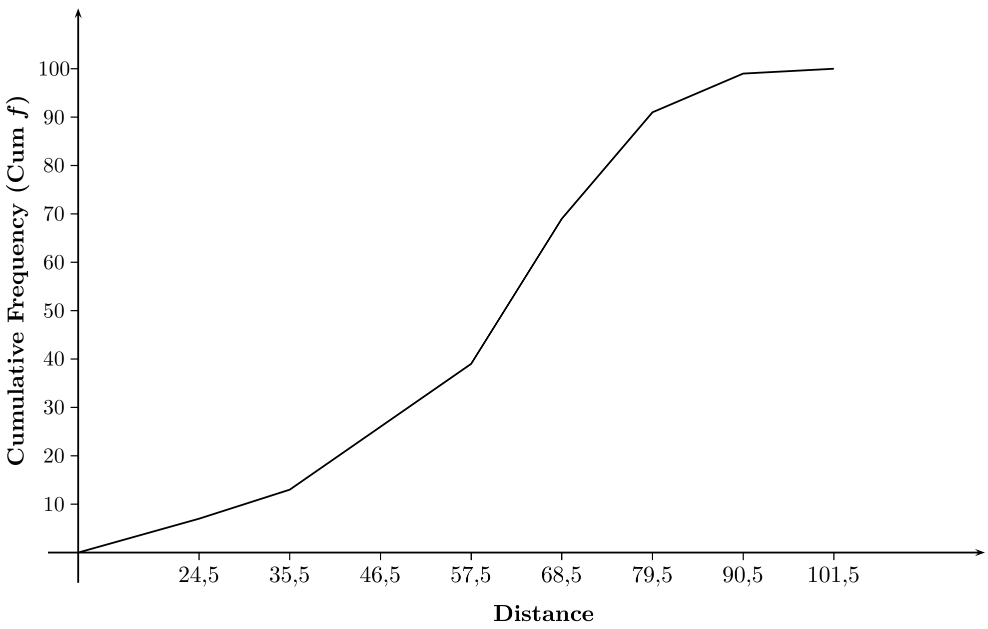


Figure 2.2.3

A very useful diagram which is easy to set up is the stem-and-leaf diagram.

The first step in setting up this diagram is to decide how to separate each observation into two parts – the stem and the leaf.

Let us separate Radial's data in such a way that the first digit of each number is the stem, and the second digit is the leaf. We already know that the smallest number is 14 and the largest is 98.

Thus 14 has stem 1 and leaf 4.
Thus 98 has stem 9 and leaf 8.

All the other numbers lie between these two and we can therefore set up the stem from 1 to 9.

Now the second digit of each number is written next to its stem.

Stem	**Leaf**	**Frequency**
1	9 6 6 4	4
2	4 9 6 2 2	5
3	8 7 4 2 2 0	6
4	2 6 5 6 6 5 5 5 0 0 8 6	12
5	8 9 9 0 9 3 1 0 4 8 0 6 8 1 4 1 4 1	18
6	1 6 4 6 4 9 6 6 7 4 4 7 1 1 2 9 2 2 2 7 6 7 4 9 1 2 4	27
7	2 5 7 0 2 8 5 2 5 7 8 0 8 7 0 2 4 4 5	19
8	8 0 3 2 6 6 0	7
9	8 0	2

To make it more readable we can sort the data for each stem.

Radial's sorted stem-and-leaf diagram follows:

Stem	**Leaf**	**Frequency**
1	4 6 6 9	4
2	2 2 4 6 9	5
3	0 2 2 4 7 8	6
4	0 0 2 5 5 5 5 6 6 6 6 8	12
5	0 0 0 1 1 1 1 3 4 4 4 6 8 8 8 9 9 9	18
6	1 1 1 1 2 2 2 2 2 4 4 4 4 4 4 6 6 6 6 6 7 7 7 7 9 9 9	27
7	0 0 0 2 2 2 2 4 4 5 5 5 5 7 7 7 8 8 8	19
8	0 0 2 3 6 6 8	7
9	0 8	2

Now turn the page on its side, and it is easy to see that most of the customers drove sixty thousand kilometres with a set of tyres.

Exercise 2.1

As the manager of an insurance claims division, you have to set up performance levels. You have asked 30 of your experienced claims processing personnel to record the number of claims that they processed during a specific week.

The following data set was collected:

Claims processed by 30 claims processors in a week

31	30	28	33	35	37
37	36	38	38	39	36
38	34	39	31	30	34
40	41	40	41	48	45
46	44	39	34	40	42

1. Display the data in the form of a histogram.
2. Use the frequency table to set up a cumulative frequency table.
3. Set up a stem-and-leaf diagram.
4. Now answer the questions.
   (a) During the past month an experienced worker has processed only 26 claims per week. Do you sense a problem? Give reasons for your answer.
   (b) Information obtained from a competitor indicates that 50% of his workers can process 36 to 39 claims per week. What is happening here?
   (c) You decide to transfer some of the workers to other divisions if you find that less than 36 claims are processed per week by half of the workers. What is you decision?

Study unit 2.3 Measures of locality

Learning objectives: *On completion of this study unit you should be able to calculate the mean, the mode and the median of a data set.*

2.3.1 The mean

Radial advertises that its XXX tyres will travel at least 65 000 kilometres before one of the four tyres will no longer meet the minimum safety requirements. What is the mean number of kilometres that can be driven with a set of XXX tyres? Radial has only the sample of 100 observations available to estimate the mean. If we consider the sample as being representative of the population we use the sample mean as an estimator of the population mean.

To obtain the sample mean we add up all the observations and divide the result by the number of observations. (This is called the arithmetic mean.)

Activity

Add up all the observations in Radial's sample and divide the sum by the number of observations.

Answer

The answer is

$$\frac{5\,828}{100} \times 100 = 58{,}28.$$

We can therefore expect a set of tyres to travel

$$58{,}28 \times 1\,000 = 58\,280$$

kilometres on average.

(*In Tutorial Letter 101 we explain how to do this using the recommended pocket calculator.*)

A formula for the mean is

$$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i,$$

where

$\overline{x}$	(read as x-bar) is the generally accepted symbol for the arithmetic mean
n	is the number of observations
$\sum$	is the Greek letter for S and means "sum" (see component 1)
x_i	represents the ith observation

The symbol $\sum_{i=1}^{n} x_i$ is just another way of writing $x_1 + x_2 + \ \ldots \ + x_n$.

The mean is the measure of locality used most often. However, sometimes it can be misleading.

Activity

Calculate the mean of 2; 3; 5 and 71 and compare it with the original values.

Answer

The mean is

$$\begin{aligned}\overline{x} &= \frac{2+3+5+71}{4} \\ &= 20{,}25.\end{aligned}$$

When a data set has a mean of 20, one intuitively expects most of the values to lie within the vicinity of 20. In this instance, however, most of the values are less than 6 and one value is an outlier of 71!

The mean is rather sensitive to outliers and may often be misleading. On its own, without any additional information, it may often lead to incorrect conclusions.

A big advantage of the arithmetic mean is that it uses all the available data.

We will later see that this is not the case for the other measures of locality.

Since the mean can be calculated exactly, it forms the basis for many advanced analyses, and is not only descriptive in nature.

2.3.2 The median

Since the mean is sensitive to extreme values and may often lead to misleading conclusions, the median is often preferred as a measure of locality.

The median is the value that divides an ordered data set into two equal parts.

The data set must be sorted in ascending order. Half or 50% of the data values will lie below or to the left of the median and 50% above or to the right.

The median is determined as follows. Given a data set of size n sorted in ascending sequence, the median (Me) is the

$$\frac{(n+1)\,\text{th}}{2}$$

value.

Activity

Determine the median of the following data sets:

1. 6; 9; 12; 12; 13; 15; 18; 24; 27
2. 2; 3; 5; 71

Answer

1. The data set is in ascending order. The number of observations is $n = 9$.

   The median is the
   $$\frac{9+1}{2} = 5\text{th}$$
   value. The median (Me) is 13.

2. The data set is in ascending order. The number of observations is $n = 4$.

   The median is the
   $$\frac{4+1}{2} = 2\frac{1}{2}\text{th}$$
   value. The $2\frac{1}{2}$th value is a value halfway between the second and third values, that is, between 3 and 5. The median is
   $$Me = \frac{3+5}{2} = 4.$$
   Therefore 50% of the data lie to the left of 4 and 50% to the right of 4.

2.3.3 The mode

The mode of a data set is that value which occurs most often. Once again consider the previous activity's first data set:

$$6; \quad 9; \quad 12; \quad 12; \quad 13; \quad 15; \quad 18; \quad 24; \quad 27$$

The mode is 12 because it occurred more often, that is twice.

In the second data set:

$$2; \quad 3; \quad 5; \quad 71$$

there is no mode because there is no value that occurs more than once.

However, the mode is not a good measure of locality. It may happen that there is no value that occurs more than any other value, or that there is more than one value with the same maximum number of occurrences. Furthermore, it has the same drawbacks as the median. The only thing in the mode's favour, is that it is easy to understand. We will not go into the detail involved in determining the mode of a frequency distribution. It is sufficient to be able to recognise the modal interval.

The modal interval is the interval with the highest frequency.

Exercise 2.2

Calculate the

1. mean
2. median
3. mode

for the following data:

$$190; \quad 104; \quad 135; \quad 314; \quad 179; \quad 175; \quad 170; \quad 146; \quad 127; \quad 131$$

Study unit 2.4 Measures of dispersion

Learning objectives: *On completion of this study unit you should*

- *calculate the variance of a data set*
- *explain the notion of a standard deviation*
- *calculate and interpret the quartile deviation of an unordered data set*
- *calculate and interpret a coefficient of variation for a data set*

2.4.1 The variance of a data set

In the previous study unit we considered the problems that occur when we work with a measure of locality only.

The arithmetic mean, even though it uses all the values in the data set, does not give much information about what the data set really looks like. What we also need is information on the spread of the data around the mean.

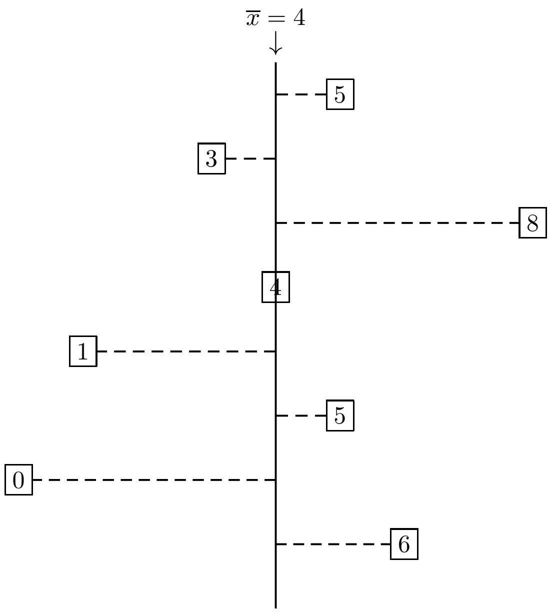
Consider the following data set:

$$5; \quad 3; \quad 8; \quad 4; \quad 1; \quad 5; \quad 0; \quad 6$$

The mean is

$$\begin{aligned} \overline{x} &= \frac{32}{8} \\ &= 4. \end{aligned}$$

Let's plot the data points around the mean.



Now calculate the distance from $\overline{x} = 4$ to each value and then calculate the mean of these distances. That is,

$$\begin{aligned} 1 + 1 + 4 + 0 + 3 + 1 + 4 + 2 &= \frac{16}{8} \\ &= 2. \end{aligned}$$

When we work with Radial's sample it may be quite difficult to calculate the mean distance. How long do you think it will take with 100 values?

An alternative is to calculate the deviation from the mean for each observation, that is, $(x - \overline{x})$

$$1; \quad -1; \quad 4; \quad 0; \quad -3; \quad 1; \quad -4; \quad 2$$

When you add this you get 0 – which tells you nothing! However, the number crunchers of the olden days did not become discouraged and came up with a clever idea.

Use $(x - \overline{x})$, but square it. The square of any value is always a positive number. The mean of the squared deviations is called the **variance**.

The positive square root of the variance is called the **standard deviation**, and we will use this measurement to give an indication of the spread of the data around the mean. The variance of a sample is defined as

$$S^2 = \frac{\sum_{i=1}^{n} (x_i - \overline{x})^2}{n - 1}.$$

Notice that we divide by $n - 1$ and not by n. The reason for this is that the sample variance is used to estimate the population variance (although we will not worry about its deduction in this course). If we were to divide by n, it would give an underestimation of the population variance. Therefore division by $n - 1$ gives a better estimator.

Now back to our sample.

	$(x - \overline{x})$	$(x - \overline{x})^2$
1	1	1
2	-1	1
3	4	16
4	0	0
5	-3	9
6	1	1
7	-4	16
8	2	4
	Total	48

The variance is

$$\begin{aligned} S^2 &= \frac{48}{8 - 1} \\ &= 6{,}86. \end{aligned}$$

The standard deviation is

$$\begin{aligned} S &= \sqrt{6{,}86} \\ &= 2{,}62. \end{aligned}$$

Activity

Calculate the standard deviation of the following sample:

Selling price of a specific share over the 14 working days (in cents)

1 630	1 550	1 430	1 440	1 390	1 400	1 480
1 490	1 410	1 905	1 540	1 890	1 900	1 900

Answer

First calculate the mean, $\overline{x}$. Then calculate the deviation from $\overline{x}$ for each observation and square it. Divide the sum of the squares by $n - 1 = 13$. The standard deviation is the square root of the variance. The mean is

$$\begin{aligned}\overline{x} &= \frac{\sum x_i}{n} \\ &= \frac{22\,355}{14} \\ &= 1\,596{,}79.\end{aligned}$$

The following table will help you with the calculations for the standard deviation:

	x_i	$(x_i - 1\,596{,}79)$	$(x_i - 1\,596{,}79)^2$
1	1 630	33,21	1 102,90
2	1 550	−46,79	2 189,30
3	1 430	−166,79	27 818,90
4	1 440	−156,79	24 583,10
5	1 390	−206,79	42 762,10
6	1 400	−196,79	38 726,30
7	1 480	−116,79	13 639,90
8	1 490	−106,79	11 404,10
9	1 410	−186,79	34 890,50
10	1 905	308,21	94 993,40
11	1 540	−56,79	3 225,10
12	1 890	293,21	85 972,10
13	1 900	303,21	91 936,30
14	1 900	303,21	91 936,30
		Total	565 180,30

The variance is

$$\begin{aligned}S^2 &= \frac{\sum_{i=1}^{14} (x_i - \overline{x})^2}{13} \\ &= \frac{565\,180{,}30}{13} \\ &= 43\,475{,}41.\end{aligned}$$

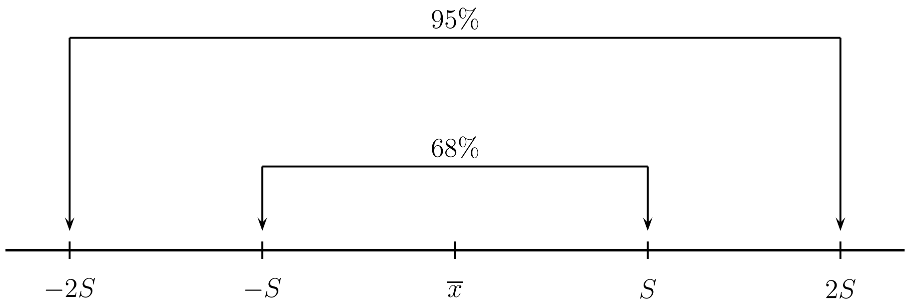
The standard deviation is

$$S = \sqrt{43\,475{,}41} = 208{,}51.$$

(*See Tutorial Letter 101 for use of the recommended pocket calculator.*)

2.4.2 The standard deviation of a data set

The standard deviation was defined as the square root of the variance. But what does it tell us? It tells us how far away the observations are from the mean. The larger the standard deviation, the further away are the data points from the mean. The following schematic presentation shows how many of the data points are between the standard deviation to the left and to the right of the mean.

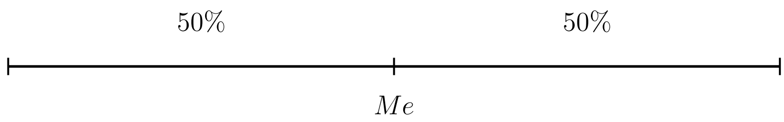


The standard deviation plays a very important role in inferential statistics, that is the field which considers the problem of making scientifically based conclusions about populations using sample data.

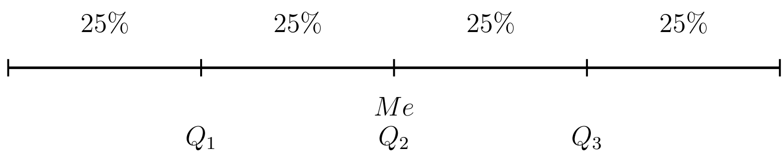
2.4.3 The quartile deviation

In study unit 6.3 you were introduced to the median (6.3.2).

The median is that value which separates a sorted data set into two equal parts.



If we divide a sorted data set into four equal sized parts we get four quartiles.



The first quartile Q_1, is the value which is the end of the first 25% of the data values; Q_2 represents the value indicating the end of the second 25% (or the value which divides the data set into two equal parts); and Q_3 is the value indicating the end of the third 25%. The second quartile Q_2, has the same value as the median.

The middle 50% of the data lies between Q_1 and Q_3. The measure

$$Q_D = \frac{Q_3 - Q_1}{2}$$

is called the quartile deviation and is the measurement of the dispersion of the data around the median.

As with the median, the quartile deviation does not use all the observations. It ignores outliers since the top 25% and the bottom 25% of the data values are not taken into account.

Activity

The purchasing manager of a group of clothing shops has recorded 15 observations on the number of days passing between reordering items from a new range of children's clothing:

Reordering intervals (in days)

17	18	26	15	17
26	23	29	28	18
22	5	12	23	22

Calculate and interpret the quartile deviation of the reordering intervals.

Answer

The value of the median or Q_2 is the value of the $\frac{1}{2}(n+1)$th observation in a ranked data set (see paragraph 6.3.2). Similarly, the value of Q_1 is the value of the $\frac{1}{4}(n+1)$th observation and the value of Q_3 is the value of the $\frac{3}{4}(n+1)$th observation in a ranked data set.

The ranked data set is

5; 12; 15; 17; 17; 18; 18; 22; 22; 23; 23; 26; 26; 28; 29

For Q_1:

$$\frac{1}{4}(15+1) = 4.$$

Therefore Q_1 is the 4th observation. Thus

$$Q_1 = 17.$$

For Q_3:

$$\frac{3}{4}(15+1) = 12.$$

Therefore Q_3 is the 12th observation. Thus

$$Q_3 = 26.$$

For Me:

$$\frac{1}{2}(15+1) = 8.$$

Therefore Me is the 8th observation. Thus

$$Me = 22.$$

The quartile deviation is

$$\begin{aligned} Q_D &= \frac{Q_3 - Q_1}{2} \\ &= \frac{26 - 17}{2} \\ &= 4{,}5. \end{aligned}$$

It can be expected that 50% of all observations will fall within 4,5 days on both sides of the median of 22 days, that is, within 17,5 and 26,5 days, or rather, 18 and 26 days. And 25% of the observations fall within 4,5 days to the left of the median value, and 25% of the observations fall within 4,5 days to the right of the median value.

2.4.4 The coefficient of variation

A stockbroker wishes to compare two unit trusts. He has the annual return rates of the two unit trusts for the past ten years available and calculates the mean and variance of each.

Fund	$\overline{x}$	S_x^2
A	16	280,34
B	12	99,37

We see that the variance for fund A is higher than that of fund B, and we draw the conclusion that the risk associated with fund A is higher than that for fund B.

Fund A, however, displayed a higher mean return over the past ten years than fund B.

This feels intuitively right – an investment having a higher associated risk should have a higher mean rate of return.

But what if fund A had a mean rate of return of 21% with the same variance? Would we then still be able to say that fund A is subject to higher fluctuation than fund B?

It is only when the two means are close together that we can compare the variances.

To get a more reliable comparison we need a measure which displays the relative variability. The coefficient of variation is such a measure and cancels the differences between means.

The coefficient of variation is

$$\begin{aligned} CV &= \frac{S}{\overline{x}} \\ &= \frac{\text{standard deviation}}{\text{mean}}. \end{aligned}$$

The coefficient of variation with the largest value has the highest relative variability.

If fund A has a mean of 16 and a standard deviation of 16,74 (ie $\sqrt{280{,}34}$), then the coefficient of variation is

$$\begin{aligned} CV(\text{A}) &= \frac{16{,}74}{16} \\ &= 1{,}05. \end{aligned}$$

For fund B the mean is 12 and the standard deviation is 9,97 (ie $\sqrt{99{,}37}$). The coefficient of variation is

$$\begin{aligned} CV(\text{B}) &= \frac{9{,}97}{12} \\ &= 0{,}83. \end{aligned}$$

We come to the same conclusion as before – the observations of fund A display a higher degree of variability than those of fund B.

But, when the mean of fund A is 21 and its standard deviation is 16,74, then the coefficient of variation is

$$\begin{aligned} CV(\text{A}) &= \frac{16{,}74}{21} \\ &= 0{,}80. \end{aligned}$$

Now fund A displays lower variability or variation than fund B.

Sometimes the coefficient of variation is multiplied by 100 to present it as a percentage.

Activity

A company that markets the seed for agricultural crops has tested three new wheat varieties and has obtained the following results:

	Type of seed		
	A	**B**	**C**
Mean number of bushels per hectare	88	56	100
Standard deviation	16	15	25

The company is not interested first and foremost in the highest yield, but rather in consistent yields.

Which type of seed would you recommend?

Answer

The coefficient of variation for seed A is

$$\begin{aligned} CV(\text{A}) &= \frac{16}{88} \\ &= 0{,}1818 \\ &= 18{,}18\%. \end{aligned}$$

The coefficient of variation for seed B is

$$\begin{aligned} CV(\text{B}) &= \frac{15}{56} \\ &= 0{,}2679 \\ &= 26{,}79\%. \end{aligned}$$

The coefficient of variation for seed C is

$$\begin{aligned} CV(\text{C}) &= \frac{25}{100} \\ &= 0{,}25 \\ &= 25\%. \end{aligned}$$

Type A displays the least variation in the yield size.

Study unit 2.5 The box-and-whiskers diagram

Learning objectives: *On completion of this study unit you should be able to present and interpret a box-and-whiskers diagram.*

In study unit 6.2 we discussed presentations that incorporate all the original data.

A very useful presentation that does not use all the data is the box-and-whiskers diagram. It uses only five observations from a data set, namely the smallest value, the first quartile, the median, the third quartile and the largest value.

Now page back to paragraph 6.4.3 on page 84 for the information on the reordering intervals for children's clothing. From the ranked data set we see that 5 is the smallest value and 29 is the largest.

We calculated that

$$Q_1 = 17, \quad Me = 22 \quad \text{and} \quad Q_3 = 26.$$

Now draw a pair of coordinate axes with the vertical axis representing the range of the observations. Draw a "box" from the first to the third quartile with the median in the middle.

Then draw lines from the box to the extreme points.

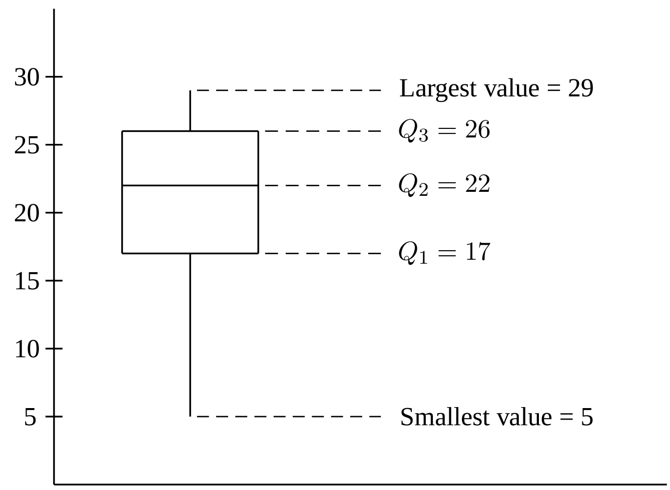


Figure 2.5.1

A box-and-whiskers diagram is especially useful when two or more distributions are compared. Also, if you are using a computer, it is not cumbersome to work with even very large data sets.

Exercise 2.3

Rainfall measurements in regions A, B and C yielded the following information:

	Region		
	A	**B**	**C**
Smallest value	0	200	0
Q_1	50	300	200
Me	200	350	800
Q_3	300	550	900
Largest value	500	1 000	950

Graphically represent the data and interpret the graphs.

COMPONENT 3

Index numbers and transformations

On completion of this component you should be able to calculate simple index numbers and perform transformations.

CONTENTS

Study unit 3.1 Index numbers

Learning objectives: *On completion of this study unit you should be able to calculate*

- *quantity indices*
- *value indices*
- *price indices*

What is an index number?

An index number can be described as a ratio, which is a measure of relative change. An index therefore represents a change in quantity, price or value.

Since a change implies a move **from** a specific point **to** another point, any index must have a starting point from where the change can be measured. This starting point is called the base. The value used for the base is written as the denominator of the ratio.

Indices are classified as simple or composite. A simple index involves one product only, while composite indices involve more than one product.

3.1.1 Price indices

The first price indices were calculated more than 200 years ago. In South Africa retail prices have been recorded and price indices calculated since 1958.

3.1.1.1 A simple price index

A simple price index is written as

$$\frac{\text{price in given year}}{\text{price in base year}} \times 100.$$

A formula for this is

$$I_n = \frac{P_n}{P_0} \times 100$$

where

I_n	is the index for the n-th period
P_n	is the price in the n-th period
P_0	is the price in the base period

Activity

In January the price of potatoes was R6,00 per one-kilogram bag. In May the price for the same quantity was R10,00. What is the price index for May if January is considered as the base?

Answer

The price index is

$$\begin{aligned} I_n &= \frac{P_n}{P_0} \times 100 \\ &= \frac{10}{6} \times 100 \\ &= 166{,}67. \end{aligned}$$

What is the meaning of a price index of 166,67? Remember that the price index represents the change from the base period to another period. The base index is 100. (Why?) Hence there was an increase of 66,67% in the price from January to May.

Index numbers are never written as percentages, even though they are calculated as percentages. (That is why 100 is included in the formula.)

An index which is greater than 100 indicates a price increase after the base period, while an index which is less than 100 indicates a decrease in price.

3.1.1.2 A composite price index

In the town of Aston an index for the cost of living is based on three items only: bread, cheese and beer. The following information is available (values in rand).

Item	2009	2012
Bread (one regular size)	5,75	10,50
Cheese (one kilogram)	49,30	54,60
Beer (one 350 ml bottle)	8,00	10,25

An index number for the three items together is

$$\begin{aligned} I &= \frac{\text{total cost of the three items in 2012}}{\text{total cost of the three items in 2009}} \times 100 \\ &= \frac{10{,}5 + 54{,}60 + 10{,}25}{5{,}75 + 49{,}30 + 8} \times 100 \\ &= \frac{75{,}35}{63{,}05} \times 100 \\ &= 119{,}51. \end{aligned}$$

Hence, there was a 19,51% increase in the cost of living. But what does this actually say?

The index implies that each of the three items carries the same weight. This may be so, but it does not sound logical. Just imagine how jolly the people from Aston would be if they were to drink beer all day! It makes more sense to take into account the quantities consumed of each product as well. In other words, the index is weighted. This brings us to the next problem – which year's quantities will we weigh?

There are actually two systems in use and both work on the assumption that the quantities being consumed do not change.

Laspeyres, the inventor of the first one, assumed that people are now still buying the same quantities as they bought in the base year. This is called the base-weighted price index.

The base-weighted price index is

$$\frac{\text{total cost of the base year quantities } (q_0) \text{ at current prices } (p_n)}{\text{total cost of the base year quantities } (q_0) \text{ at base year prices } (p_0)} \times 100.$$

Laspeyres price index is calculated as

$$P_L(n) = \frac{\sum p_n q_0}{\sum p_0 q_0} \times 100$$

where

p_n	is the price in the n-th year
p_0	is the price in the base year
q_0	is the quantity in the base year

The alternative index, that of Paasche, uses the quantities of the year for which the index is calculated.

The current-weighted price index is

$$\frac{\text{total cost of current quantities } (q_n) \text{ at current prices } (p_n)}{\text{total cost of current quantities } (q_n) \text{ at base year prices } (p_0)} \times 100.$$

In this case

$$P_P(n) = \frac{\sum p_n q_n}{\sum p_0 q_n} \times 100$$

where

p_n	is the price in the n-th year
p_0	is the price in the base year
q_n	is the quantity in the n-th year

Activity

Calculate both the Laspeyres and the Paasche price indices for 2012 with 2009 as base year using the following data of the Williams family of Aston. Interpret the indices.

Item	Quantity 2009	Price 2009	Quantity 2012	Price 2012
Bread (one regular size)	280	5,75	300	10,50
Cheese (one kilogram)	90	49,30	70	54,60
Beer (one 350 ml bottle)	400	8,00	600	10,25

Answer

The 2009 prices and quantities will be indicated by p_0 and q_0, while the 2012 prices and quantities will be indicated by p_n and q_n.

The best way of explaining the calculations is by way of a table.

2009		2012					
p_0	q_0	p_n	q_n	$p_0 \times q_0$	$p_0 \times q_n$	$p_n \times q_0$	$p_n \times q_n$
5,75	280	10,50	300	1 610	1 725	2 940	3 150
49,30	90	54,60	70	4 437	3 451	4 914	3 822
8,00	400	10,25	600	3 200	4 800	4 100	6 150
				9 247	9 976	11 954	13 122

The Laspeyres price index is

$$\begin{aligned} P_L(n) &= \frac{\sum p_n q_0}{\sum p_0 q_0} \times 100 \\ P_L(2012) &= \frac{\sum p_{2012} q_{2009}}{\sum p_{2009} q_{2009}} \times 100 \\ &= \frac{11\,954}{9\,247} \times 100 \\ &= 129{,}27. \end{aligned}$$

The Paasche price index is

$$\begin{aligned} P_P(n) &= \frac{\sum p_n q_n}{\sum p_0 q_n} \times 100 \\ P_P(2012) &= \frac{\sum p_{2012} q_{2012}}{\sum p_{2009} q_{2012}} \times 100 \\ &= \frac{13\,122}{9\,976} \times 100 \\ &= 131{,}54. \end{aligned}$$

The Laspeyres index: For the same quantity of goods bought in 2009, the price paid in 2012 is 29,27% higher.

The Paasche index: The same quantity goods as bought in 2012 would have cost 31,54% less in 2009.

And now for the big question: which index should be used?

The biggest disadvantage of Laspeyres is that historic quantities are used which represent spending patterns that may no longer be valid. However, when the quantities currently bought are not that much different, it does not really matter.

A big advantage, however, is that we do not need to know what the current spending patterns are to calculate the index. As soon as new prices become known, the index can be calculated.

The biggest advantage is that the denominator, the $\sum p_0 q_0$ term, remains the same. It therefore has to be calculated only once. It may sound ridiculous that this is important, but the consumer price index is based on a basket of goods.

Of further importance is the fact that indices calculated using the same denominator are directly comparable. The denominator of the Paasche index, $\sum p_0 q_n$, changes annually. Indices calculated in this way may only be compared with the base year index and not directly with each other. Moreover, one has to wait until the end of a specific year to calculate the current quantities for that year.

It therefore appears as if the Paasche index has more disadvantages than advantages, except in cases where the quantities change substantially from one period to the next.

3.1.2 Quantity indices

A quantity index measures the change that is attributable to a change in the quantities bought. A quantity index may also be described as a volume index.

As with the price indices, we use Laspeyres and Paasche indices, with Laspeyres once again being the favourite.

The Laspeyres quantity index is calculated using the total quantities of a basket of goods used in the base year.

The base-weighted quantity index is

$$\frac{\text{total quantities of current year } (q_n) \text{ at base year prices } (p_0)}{\text{total quantities of base year } (q_0) \text{ at base year prices } (p_0)} \times 100.$$

Laspeyres quantity index is calculated as

$$Q_L(n) = \frac{\sum p_0 q_n}{\sum p_0 q_0} \times 100.$$

Similarly, the Paasche quantity index is

$$\frac{\text{total quantities of current year } (q_n) \text{ at current prices } (p_n)}{\text{total quantities of base year } (q_0) \text{ at current prices } (p_n)} \times 100.$$

In this case

$$Q_P(n) = \frac{\sum p_n q_n}{\sum p_n q_0} \times 100.$$

Activity

Use the information given for the previous activity to calculate the Laspeyres and Paasche quantity indices for 2012, with 2009 as the base year for the Williams household of Aston.

Answer

We use the same table as in the previous activity.

2009		2012					
p_0	q_0	p_n	q_n	$p_0 \times q_0$	$p_0 \times q_n$	$p_n \times q_0$	$p_n \times q_n$
5,75	280	10,50	300	1 610	1 725	2 940	3 150
49,30	90	54,60	70	4 437	3 451	4 914	3 822
8,00	400	10,25	600	3 200	4 800	4 100	6 150
				9 247	9 976	11 954	13 122

The Laspeyres quantity index is calculated as

$$\begin{aligned} Q_L(n) &= \frac{\sum p_0 q_n}{\sum p_0 q_0} \times 100 \\ Q_L(2012) &= \frac{\sum p_{2009} q_{2012}}{\sum p_{2009} q_{2009}} \times 100 \\ &= \frac{9\,976}{9\,247} \times 100 \\ &= 107{,}88. \end{aligned}$$

The Paasche quantity index is calculated as

$$\begin{aligned} Q_P(n) &= \frac{\sum p_n q_n}{\sum p_n q_0} \times 100 \\ Q_P(2012) &= \frac{\sum p_{2012} q_{2012}}{\sum p_{2012} q_{2009}} \times 100 \\ &= \frac{13\,122}{11\,954} \times 100 \\ &= 109{,}77. \end{aligned}$$

3.1.3 Value indices

A value index is a ratio which indicates how much the value of expenditure has changed as a ratio of expenditure in the base year. It gives the "value" of a basket of goods in a specific year as a percentage of the value of that same basket in the base year:

$$V = \frac{\sum p_n q_n}{\sum p_0 q_0} \times 100.$$

Activity

Calculate the value index for the Williams household of Aston for 2012, with 2009 as the base year.

Answer

The value index is

$$\begin{aligned} V &= \frac{\sum p_n q_n}{\sum p_0 q_0} \times 100 \\ &= \frac{\sum p_{2012} q_{2012}}{\sum p_{2009} q_{2009}} \times 100 \\ &= \frac{13\,122}{9\,247} \times 100 \\ &= 141{,}91. \end{aligned}$$

The increase in the value of the basket of goods bought by the Williams household is 41,91%.

3.1.4 The consumer price index

The consumer price index has been used since 1958.

The consumer price index is an index based on a basket containing a large quantity of goods. It includes items like the food, housing, recreation and health expenditure of a "typical" or "average income" household.

However, there are a few problems with this index:

What are the most appropriate items to be included in the basket?
What are the most appropriate weights with which to weigh the items?
What is the most realistic base year?

The consumer price index, which is generally known as the CPI, is an adjusted version of the Laspeyres price index. It is calculated monthly and is used as a measure of inflation. However, all prices are not measured monthly. Some are measured every three months on a staggered basis (eg cigarettes and alcoholic beverages in January, new cars in February and textiles and furniture in March). In addition, prices for housing and education are collected annually. These facts should be borne in mind when interpreting the CPI, because they tend to result in stepped increases. The base year is adjusted every five years to ensure that the basket of goods remains representative. The latest base year is 2010.

The CPI includes indirect tax (such as VAT), while direct tax is excluded.

One of the applications of the CPI is for adjusting prices, wages, salaries and other variables for changes in the inflation rate.

Activity

The following figures for the CPI are available for 2012. Calculate the average CPI for 2012.

2011	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
CPI	108,9	108,6	107,9	107,7	107,1	107,3	104,9	103,2	101,9	101,7	101,9	102,2

Answer

The average CPI for 2012 is

$$\begin{aligned}\overline{x} &= \frac{1}{12}\sum_{i=1}^{12} x_i \\ &= \frac{1\,263{,}3}{12} \\ &= 105{,}30.\end{aligned}$$

Exercise 3.1

The quantities and prices of three products consumed over the past four years in Riverside, a neighbouring town of Aston, are given in the following table:

	2009	2010	2011	2012
Meat:				
Quantity (kg)	200	150	250	200
Price (R/kg)	34,50	39,20	36,50	41,00
Maize:				
Quantity (kg)	1 000	2 000	2 000	2 200
Price (R/kg)	10,20	10,50	10,00	11,75
Milk:				
Quantity (l)	365	400	460	550
Price (R/l)	9,50	9,75	10,50	10,00

1. Calculate the Laspeyres and Paasche price indices for 2012, with 2009 as the base year.
2. Calculate the Laspeyres and Paasche quantity indices for 2012, with 2010 as the base year.
3. Calculate the value index for 2012, with 2009 as the base year.

Study unit 3.2 Transformations and rates

Learning objectives: *On completion of this study unit you should*

- *calculate the purchasing power in a specific period*
- *calculate an exchange rate*
- *do transformations and calculations with a fine ounce of gold*
- *calculate an unweighted growth rate*

3.2.1 Uses of the consumer price index (CPI)

Consider the following table:

Year	Column 1 Base year = 1994 (1994 = 100)	Column 2 Base year = 1970 (1970 = 100)	Column 3 Base year = 2008 (2008 = 100)
1970	29,6	100,0	18,1
1971	29,9	101,0	18,3
1972	30,2	102,0	18,5
1994	100,0	337,8	61,3
1995	107,6	363,5	66,0
1996	109,6	370,3	67,2
2007	160,6	542,6	98,5
2008	163,1	551,0	100,0
2009	166,6	562,8	102,1

Table 5.2.1

3.2.1.1 Comparing costs of living

Column 1

Column 1 of table 5.2.1 represents the CPI computed using 1994 as the base year. Notice that in the base year the index has a value of 100. The CPI for other years in column 1 represent the way consumer cost of living has changed from the base year. For example, in 2008 the CPI is 163,1. This means that the cost of living has increased, and in 2008 it took R163,10 to buy what R100 would have bought in the base year, 1994. This also means that the cost of living in 2008 was 63,1% (163,1 − 100) higher than in 1994. The CPI in 2008 (163,1), minus the CPI in the base year (100), is the percentage change in prices (63,1%) between the base year and 2008. Please note that this type of calculation can only be done between the base year and some other year, not between any other two years. For example, the difference in the CPI between 2008 and the CPI in 1995 (163,1 − 107,6) does not represent the percentage change in prices between 1995 and 2008.

3.2.1.2 Change of base year

Notice that the data in column 2 of table 5.2.1 where the CPI is computed with 1970 as the base year (1970 = 100), is different from the data for column 1 with 1994 as the base year. We convert the base year from 1994 to 1970 by dividing each CPI number in column 1 by the value of the CPI in 1970 of column 1 and list the results in column 2:

Column 2

$$\begin{aligned}
\text{CPI for 1970: } & \frac{29{,}6}{29{,}6} \times 100 = 100{,}0 \\
\text{CPI for 1971: } & \frac{29{,}9}{29{,}6} \times 100 = 101{,}0 \\
& \vdots \\
\text{CPI for 1994: } & \frac{100{,}0}{29{,}6} \times 100 = 337{,}8 \\
& \vdots \\
\text{CPI for 2009: } & \frac{166{,}6}{29{,}6} \times 100 = 562{,}8.
\end{aligned}$$

Now the cost of living in various years can be compared to the cost of living in 1970 by subtracting the CPI of 100 in 1970 from the CPI for any other year. For example, the cost of living increased by 451% (551 − 100) between 1970 and 2008. One could also say that it took R551 in 2008 to buy what R100 would have bought in 1970.

Similarly, we convert the base year from 1970 to 2008 by dividing each CPI number in column 2 by the value of the CPI in 2008 of column 2 and list the results in column 3:

Column 3

$$\begin{aligned}
\text{CPI for 1970: } & \frac{100{,}0}{551{,}0} \times 100 = 18{,}1 \\
& \vdots \\
\text{CPI for 2008: } & \frac{551{,}0}{551{,}0} \times 100 = 100{,}0 \\
& \vdots \\
\text{CPI for 2009: } & \frac{562{,}8}{551{,}0} \times 100 = 102{,}1.
\end{aligned}$$

3.2.1.3 Deflating wages

The CPI can be used to deflate wages (convert to real wages) earned at different points in time. Consider a woman's income in 2007 as R145 000 and consider that same woman's income in 2009 of R172 000. Did her **real** income increase by R27 000 (172 000 − 145 000)?

Only in nominal terms, not in real terms. In other words, her income in actual buying power did not rise by R27 000. To find out how much her buying power actually increased or decreased in real buying power, we must deflate (convert to real wages) the income using the CPI. We do this by dividing the income earned in 2007 by the CPI in 2007 and also divide the income earned in 2008 by the CPI in 2008 and then compare the incomes. For this example, we will use the 2008 base year (column 3). That is, we will measure incomes in 2008 rands, not in 2007 or 2009 rands.

For 2007:

$$\begin{aligned}
\text{real income} &= \frac{145\,000}{98{,}5} \times 100 \\
&= 147\,208{,}12.
\end{aligned}$$

Her real income for 2007 in terms of 2008 rands is R147 208,12.

For 2009:

$$\begin{aligned}\text{real income} &= \frac{172\,000}{102{,}1} \times 100 \\ &= 168\,462{,}29.\end{aligned}$$

Her real income for 2009 in terms of 2008 rands is R168 462,29.

The change in income was R21 254,17 (168 462,29 – 147 208,12). Notice that the buying power of her money increased by less than R27 000 from 2007 to 2009.

This increase can also be expressed in terms of percentages. The percentage increase in terms of **nominal** values is

$$\frac{172\,000 - 145\,000}{145\,000} \times 100 = 18{,}62\%,$$

while the percentage increase in terms of **real** values is

$$\frac{168\,462{,}29 - 147\,208{,}12}{147\,208{,}12} \times 100 = 14{,}44\%.$$

Therefore, the consumer price index is used to adjust prices to eliminate the effect of inflation.

Inflation is a word that we hear a lot. Inflation is a sustained increase in the general price level. It is measured by using a price index, usually the CPI, as a proxy for the price level.

Deflating is the opposite of inflating.

Thus, to calculate a deflated value, the effect of inflation must be eliminated. This is done by dividing the value being considered in a specific year by the CPI for that year and then multiplying the answer by 100. It is also important to specify which year is the base year, in other words, the year for which real money values will be compared.

Let's see again how it is done.

Activity

A pensioner's monthly pension was R870 in June 2009 and R1 250 in January 2011. The CPI was 104,1 in June 2009 and 109,0 in January 2011. Did the purchasing power of his money increase? (The base month is January 2009.)

Answer

Consider the following table:

Month	Pension (in rand)	CPI	Deflated wage at January 2009 prices (R)
June 2009	870	104,1	$\frac{870}{104{,}1} \times 100 = 835{,}73$
January 2011	1 250	109,0	$\frac{1\,250}{109{,}0} \times 100 = 1\,146{,}79$

Although he gets R380 (ie 1 250 − 870) more per month, his purchasing power has only increased by R311,06 (ie 1 146,79 − 835,73).

3.2.2 Exchange rate

An exchange rate is the price of one country's currency in terms of another country's currency.

The foreign exchange market (forex market) is a market in which foreign currencies are exchanged for one another.

In South Africa, for example, there is a market for American dollars (US$). This means that there is a demand and a supply of US$. South Africa's export to the USA earns dollars; it creates a supply of dollars in the South African market. South Africa wishes to import goods from the USA; this represents the demand for dollars. The market price is the R/$ exchange rate which is brought about in the market.

Usually direct quotation is used, that is, R per $ in South Africa. Exchange rates vary constantly and are published daily by the media.

Activity

What is the current exchange rate of the rand against the US dollar?

What does this mean?

Devaluation occurs when the exchange rate of a country's currency is held at fixed rates of other important foreign currencies. When a more flexible exchange rate is in use, the currency values are not fixed but are determined by market forces. Then a decrease in the value of a currency is called depreciation.

The free market value of a currency is determined by the interaction between supply and demand.

3.2.3 Fine ounce

In March of 1968 the fixed gold price was abolished. Since then the gold price has been an important economic indicator in South Africa.

The price of gold is quoted in US dollars per fine ounce. A fine ounce is equal to 31,10348 grams.

The gold price is determined by supply and demand on the international gold market. The international gold market operates around the clock through the different time zones from Tokyo to Hong Kong, Zurich, Frankfurt, London and New York. The gold price is "fixed" every weekday in London at 10:30 and 15:00 at a meeting of representatives of the five member companies of the London gold market. Although many different prices can prevail during any 24-hour period, it is usually the London fixings that are reflected in contracts and official statistics.

The South African gold market is dominated by the South African Reserve Bank. Except for a relatively small percentage that is used to produce Kruger rands and one kilogram bars, the whole output of the gold mines has to be sold to the Reserve Bank. The Reserve Bank is the sole seller of gold bullion in the international gold market.

Activity

How many fine ounces of gold are there in one kilogram of gold?

Answer

If 31,10348 grams equal 1 fine ounce, then 1 000 grams equal

$$\frac{1\,000}{31{,}10348} = 32{,}1507.$$

There are 32,1507 fine ounces in one kilogram of gold.

3.2.4 Growth rate

A rate of change, or a growth rate, is the percentage change in a variable between two dates or periods. It is usually expressed as an annual rate.

The growth rate of the economy is usually measured in terms of the growth of the real gross domestic product called the real GDP.

The GDP is the total value of all final goods and services produced in the economy in a given period.

Since it is impossible to add together the physical output of the large variety of goods and services produced in the economy, the GDP necessarily has to be measured in money terms. The nominal or current price GDP is not a suitable base for measuring economic growth, since it also reflects increases in prices. To calculate the growth rate, the nominal GDP must first be adjusted for price changes.

The real GDP is the total value of all final goods and services produced within the borders of the country in a specific period (usually a year) measured at constant prices. This means that the prices ruling in a specific base year are used. We will use the real GDP to calculate a growth rate.

There is a large range of growth rates, but we will only consider the unweighted growth rate.

The real GDP for 2003 is 449 304 and the real GDP for 2011 is 558 760.

We wish to calculate percentage growth from 2003 to 2011 – a period of 8 years.

The percentage change from 2003 to 2011 is calculated as follows:

$$\begin{aligned}\left[\left(\frac{BBP_n}{BBP_0}\right)^{\frac{1}{n}} - 1\right] \times 100 &= \left[\left(\frac{BBP_{2011}}{BBP_{2003}}\right)^{\frac{1}{8}} - 1\right] \times 100 \\ &= \left[\left(\frac{558\,760}{449\,304}\right)^{\frac{1}{8}} - 1\right] \times 100 \\ &= 2{,}76\% \text{ per year.}\end{aligned}$$

Activity

The real GDP for 2005 is 461 656 and the real GDP for 2011 is 545 365. What is the growth rate from 2005 to 2011?

Answer

The period is 6 years, and the growth rate is

$$\left[\left(\frac{BBP_{2011}}{BBP_{2005}}\right)^{\frac{1}{6}} - 1\right] \times 100 = \left[\left(\frac{545\,365}{461\,656}\right)^{\frac{1}{6}} - 1\right] \times 100$$
$$= 2{,}82\% \text{ per year.}$$

(*See Tutorial Letter 101: Using the recommended calculator.*)

Although the GDP is an indispensable indicator of a country's economic performance, it tells only part of the story. A larger physical flow of goods and services does not necessarily increase the national wellbeing. In this respect the following has to be borne in mind:

- Unwanted by-products such as crime, pollution, traffic congestion, noise and psychological stress are not measured.
- The GDP figures do not distinguish between different types of production or expenditure. R100 million spent on military equipment is regarded in exactly the same light as R100 million spent on health or education.
- No allowance is made for the exhaustion of scarce resources.
- It is difficult to account for changes in the quality of goods and services.

Exercise 3.2

1. In 2008, the CPI was 97,08 and in 2010 the CPI was 100,0 (2010 is the base year). A labourer's monthly earnings were R5 050,00 in 2008 and R6 020,00 in 2010. Did the purchasing power of his money increase?
2. The R/$ (rand/dollar) exchange rate is

   $$\text{R}8{,}12 = \$1{,}00.$$

   How many US dollars can I buy for R4 000,00?
3. The price of gold is $933,80 per fine ounce. The exchange rate is

   $$\text{R}8{,}12 = \$1{,}00.$$

   What is the value in rand of one kilogram of gold?
4. The real GDP for 2008 is 517 178 and the real GDP for 2011 is 558 760. What is the growth rate from 2008 to 2011?

COMPONENT 4

Functions and representations of functions

On completion of this component you should be able to

- explain the concept of a function
- differentiate among linear, quadratic, exponential and logarithmic functions
- represent the functions graphically

CONTENTS

Study unit 4.1 What is a function?

Learning objectives: *On completion of this study unit you should be able to explain the concepts of a formula and a function.*

4.1.1 Formulae

Expressions like

$$A \times B, \;\; A \times B + A \times C, \;\; A^2 + B^2, \;\; A^{M+N}$$

which do not contain numbers only but also variables, are referred to as algebraic expressions, in contrast to the arithmetic expressions which contain only numbers. In general, we will not, however, make this subtle distinction and simply refer to expressions.

A **formula** is a specific algebraic expression which results when symbols or letters are used to represent the **relationship between different variables** in a concise and clear way. It is in fact a recipe for calculating the value of some desired variable, called the **dependent variable**, from the values of the relevant **independent** variables.

Actually, you are well acquainted with the use of formulae although, possibly, you may not generally resort to the use of symbols but use words instead. An example of a "word" formula is

bank balance at end of month = bank balance at beginning of month + sum of all transactions during the month.

Using symbols or letters we could write

$$B_e = B_b + T.$$

In this example B_e is the dependent variable and B_b and T are the independent variables. This is so because the value of B_e is determined by the values of B_b and T.

Please note, there is no binding reason to use these specific symbols and we could in fact have written

$$A = B + C.$$

The calculation of a worker's wage is

$$\text{wage (in rands)} = \text{hours worked} \times \text{rate (in rand per hour)}.$$

Using obvious symbols we write

$$W = H \times R.$$

Here W is the dependent variable, and H and R are the independent variables.

From these two examples alone, it is obvious that a major advantage of formulae is their brevity. Following from this is the fact that they may be easily manipulated.

4.1.2 The concept of a function

When the value of one variable is dependent on the value of several others, we make the statement that the dependent variable is a **function** of the independent variable.

The way in which the value of the dependent variable is determined from the values of the independent variables must be clearly stated and unambiguous, and only one single value for the dependent variable must result.

The worker's wage was written as

$$W = H \times R.$$

W is a function of H and R and is dependent on the values of H and R.

Activity

Consider the formula

$$P = \frac{I}{RT}$$

which is obtained from the basic simple interest formula by rearrangement. Identify the dependent and the independent variables in this formula.

Answer

The dependent variable is P and I, R and T are the independent variables.

The term "function of" occurs so often in mathematics that there is a special notation to denote it, namely $f(\ldots)$. The letter f obviously stands for function, while all the independent variables are listed in the brackets, separated by commas or semicolons. In this notation the function

$$S = P(1 + RT)$$

is written as

$$S = f(P;\ R;\ T)$$

which is read: S is a function of P, R and T.

Take note, however, that with this notation no information with regard to the specific form (that is rule applicable) is conveyed. It is merely stated that S is the dependent variable which is a function of P, R and T. In this sense the notation does not discriminate between the above function and any other function of P, R and T.

The function is only completely specified once the **functional form** or **relationship**, that is, the rule for determining the value of the dependent variable from the values of the independent variables, is given.

Thus, a complete specification of the function $S = P \times (1 + RT)$ is

$$S = f(P;\ R;\ T) = P \times (1 + RT).$$

Why then do we use this notation if it does not convey complete information about the function? The answer is that the notation provides us with a concise and clear way of indicating that, in a particular function, which has been previously defined, particular numerical values are to be substituted for the independent variables.

Suppose we consider the function

$$S = f(P;\ R;\ T) = P \times (1 + RT)$$

$f(1;\ 2;\ 3)$ means that 1 is substituted for P, 2 for R and 3 for T, and its value is

$$f(1;\ 2;\ 3) = 1(1 + 2 \times 3) = 7.$$

Similarly

$$f(1;\ 1;\ 1) = 1 \times (1 + 1 \times 1) = 2,$$
$$f(2\,000;\ 0{,}1;\ 3) = 2\,000 \times (1 + 0{,}1 \times 3) = 2\,600,$$
$$f(350;\ 0{,}08;\ 2{,}5) = 350 \times (1 + 0{,}08 \times 2{,}5) = 420,$$

and so on.

Note: Where confusion may arise with the decimal, we use a semicolon (;) to separate the variables.

Activity

Calculate the following values of the function

$$S = f(P;\ R;\ T) = P \times (1 + RT)$$

1. $f(10\,000;\ 0{,}1;\ 10)$
2. $f(1\,500;\ 0{,}075;\ 4)$

Answer

1. The function is

$$\begin{aligned} f(10\,000;\ 0{,}1;\ 10) &= 10\,000 \times (1 + 0{,}1 \times 10) \\ &= 20\,000. \end{aligned}$$

2. The function is

$$\begin{aligned} f(1\,500;\ 0{,}075;\ 4) &= 1\,500 \times (1 + 0{,}075 \times 4) \\ &= 1\,950. \end{aligned}$$

Study unit 4.2 Linear functions

Learning objectives: *On completion of this study unit you should be able to make a graphical representation of a linear function.*

4.2.1 The set of axes

If we want to represent the linear function $y = f(x)$ graphically, then $y = f(x)$ means that y is a function of x. The dependent variable is y and the independent variable is x.

To represent a linear function graphically, we use a set of axes.

Draw two lines at right angles to each other as shown in figure 4.2.1. The **horizontal line** is generally used for the **independent variable** and the **vertical** for the **dependent variable**. The point where they cross is the common origin. A convenient scale, which need not be the same for both lines, is indicated on each. Just as it is customary to associate points to the right of the origin with positive values of the independent variable, and points to the left with negative values, so it is customary to associate points above the origin with positive values and points below it with negative values of the dependent variable. Furthermore, it is customary to refer to the horizontal line as the ***x*-axis** and the vertical line as the ***y*-axis**. A scaled set of axes introduced in this way is referred to as a **rectangular coordinate system**. As indicated in the figure, this divides the plane (that is the sheet of paper) into four sections which are known as **quadrants** and which are numbered as shown.

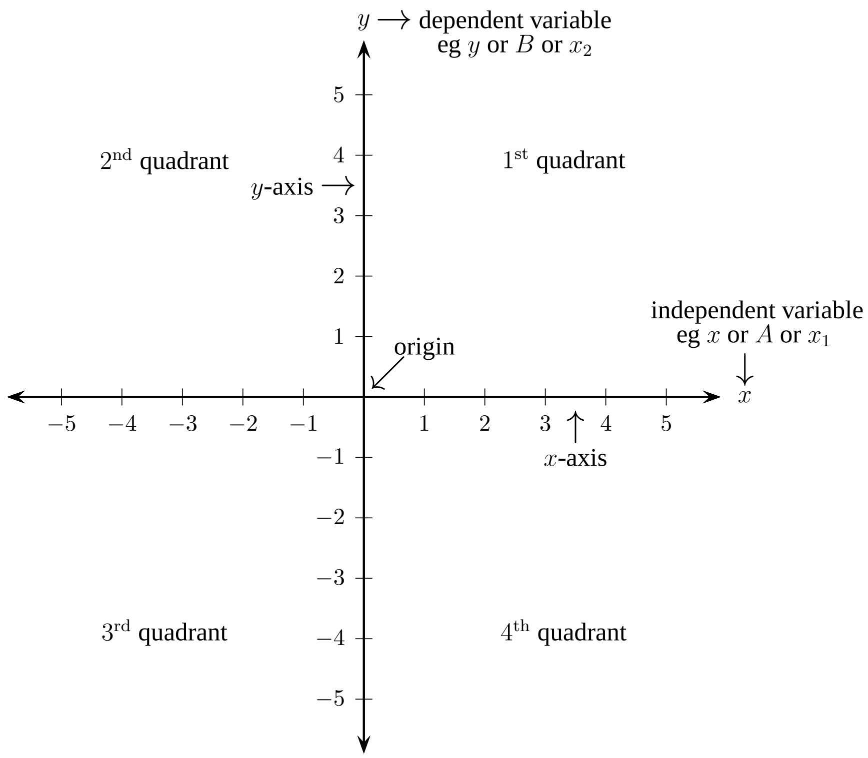


Figure 4.2.1

If x is the independent variable and y the dependent variable, then x and y are both positive in the first quadrant. In the second quadrant x is negative and y is positive, in the third quadrant x and y are both negative, and in the fourth quadrant x is positive and y is negative. Since most business problems deal with positive quantities, we shall be concerned mainly with points in the first quadrant, but if we regard losses as negative profits, deductions as negative additions, deficits as negative income, and so on, we shall also have the occasion to work with points in the other three quadrants. Whenever "reading" a graph, you carefully establish the variables represented on each axis and the relevant scales. **Always label the axes clearly**. Remember, the variables you want to draw on the axes need not be x and y, but any variables, for example A and B or x_1 and x_2.

Now, according to the definition of a function, a function assigns one value of y to each value of x within its domain. Thus a set of ordered pairs of data, which we can write as $(x; y)$, are established. Each of these pairs corresponds to a point in the plane, and if we plotted all these points we would obtain what is called a graph of a given function.

In other words, the graph of the function $y = f(x)$ consists of all ordered pairs $(x; y)$ which satisfy $y = f(x)$. We speak of the **coordinates** $(x; y)$ of each point P, and call x the **abscissa** and y the **ordinate** of P.

Of course, it is not only points on the graph of $y = f(x)$ which may be referred to in this way. Any point in the plane is located by the specification of an ordered pair of numbers $(x; y)$. That is in fact why we refer to a rectangular **coordinate system**.

We start by considering the **straight line** or **linear function**, as it is known, which is the most elementary, and perhaps the most important, of all functions.

The general functional expression of a straight line is

$$y = f(x) = ax + b$$

where a and b are constants.

4.2.2 The intercept of a straight line

Let us look at the general properties of a straight line.

The point where the **line cuts the *y*-axis**, is called the ***y*-intercept**. This is where $x = 0$. At this point the value of y is

$$\begin{aligned} y &= a \times 0 + b \\ &= b. \end{aligned}$$

In other words, we say that the **intercept on the *y*-axis** is equal to the constant term ***b*** in the functional expression for the straight line.

The point where the **line cuts the *x*-axis** is called the ***x*-intercept** or **root**. The value of the **intercept on the *x*-axis** is where

$$y = 0,$$

this is

$$ax + b = 0.$$

We can easily solve this to obtain the value of x:

$$ax + b - b = -b$$

this is

$$ax = -b$$

(since $+b - b = 0$).

Next, assuming that a is not 0, we divide by a:

$$\frac{ax}{a} = -\frac{b}{a}$$

this is

$$x = -\frac{b}{a}.$$

See paragraph 2.2.6, **Special cases**, for the discussion when $a = 0$.

This is the value of the intercept on the x-axis.

Thus, we can conclude that the line cuts the axes at the points $(0; b)$ and $\left(-\frac{b}{a}; 0\right)$ as shown in figure 4.2.2.

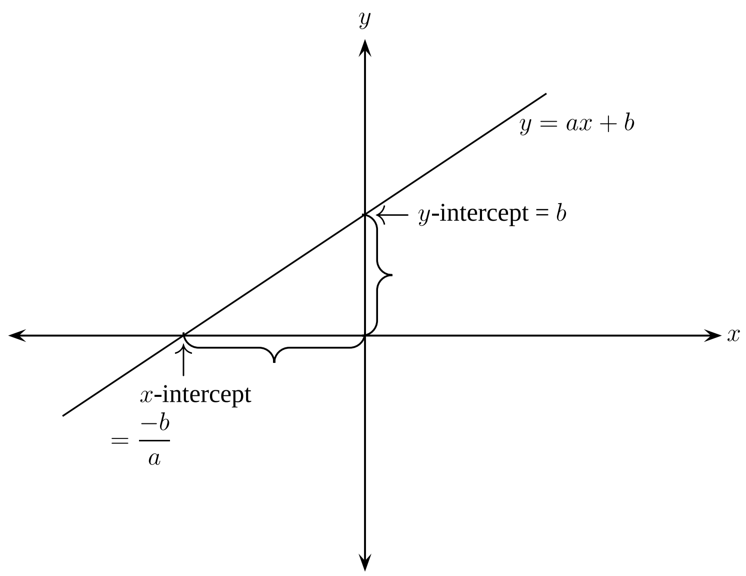


Figure 4.2.2

Note that in this figure a and b are assumed to be positive.

The four specific cases which can occur are illustrated in figure 4.2.3.

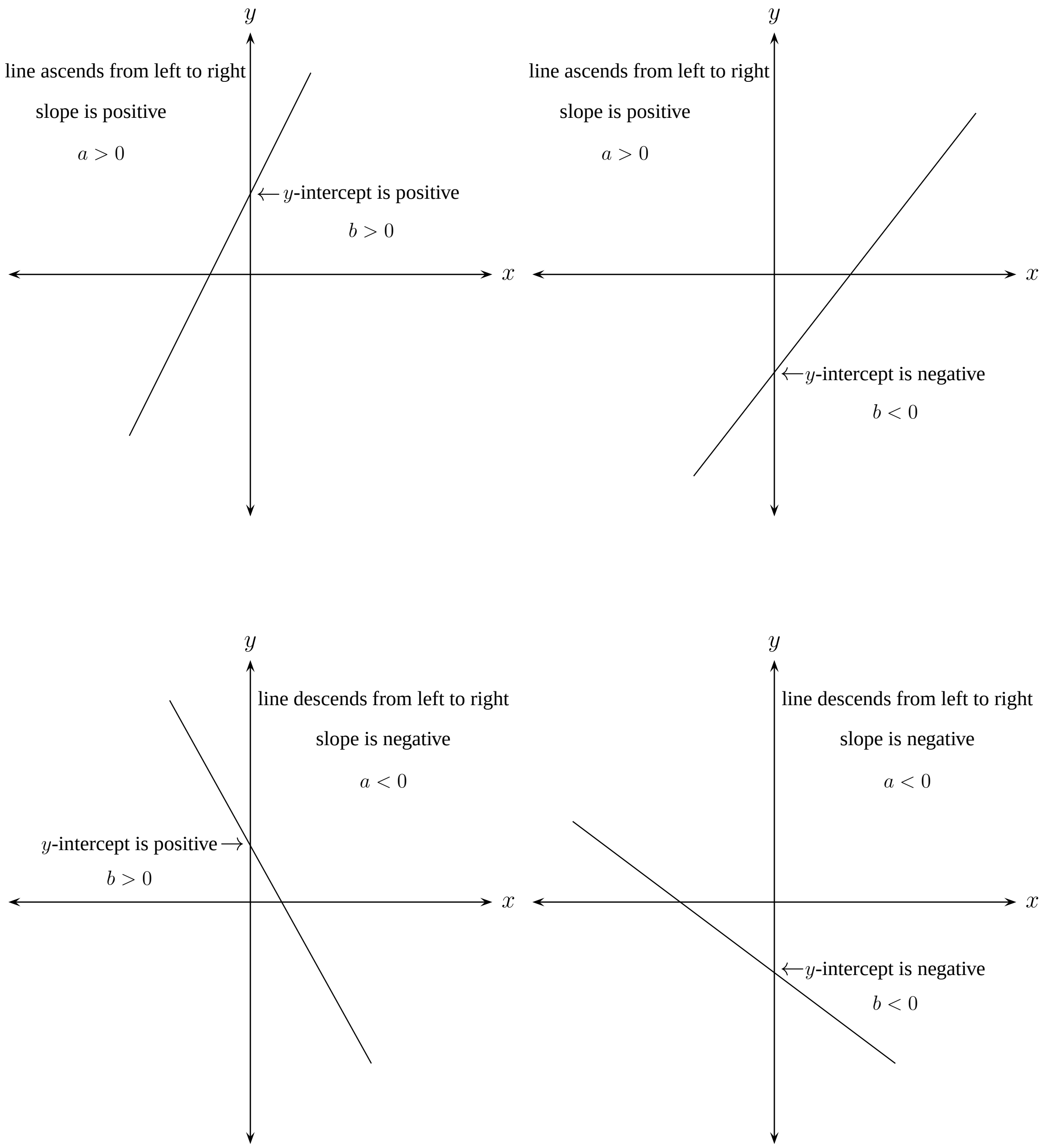


Figure 4.2.3

Activity

Determine the intercepts on the x- and y-axes of the following straight lines:

1. $y = 1 + x$
2. $y = 2 - 4x$
3. $y = -6 + 9x$
4. $y = -25 - 5x$

Answer

The intercepts on the axes are as follows:

1. Determine the y-intercept: if $x = 0$, then

$$\begin{aligned} y &= 1 + 0 \\ &= 1. \end{aligned}$$

The coordinates are $(0; 1)$.

Determine the x-intercept: if $y = 0$, then

$$\begin{aligned} 0 &= 1 + x \\ x &= -1. \end{aligned}$$

The coordinates are $(-1;\ 0)$.

2. Determine the y-intercept: if $x = 0$, then

$$\begin{aligned} y &= 2 - 4(0) \\ &= 2. \end{aligned}$$

The coordinates are $(0; 2)$.

Determine the x-intercept: if $y = 0$, then

$$\begin{aligned} 0 &= 2 - 4x \\ -2 &= -4x \\ -4x &= -2 \\ \frac{-4x}{-4} &= \frac{-2}{-4} \\ x &= \frac{1}{2}. \end{aligned}$$

The coordinates are $\left(\frac{1}{2};\ 0\right)$.

3. Determine the y-intercept: if $x = 0$, then

$$\begin{aligned} y &= -6 + 9(0) \\ &= -6. \end{aligned}$$

The coordinates are $(0; -6)$.

Determine the x-intercept: if $y = 0$, then

$$\begin{aligned} 0 &= -6 + 9x \\ 6 &= 9x \\ 9x &= 6 \\ \frac{9x}{9} &= \frac{6}{9} \\ x &= \frac{2}{3}. \end{aligned}$$

The coordinates are $\left(\frac{2}{3}; 0\right)$.

4. Determine the y-intercept: if $x = 0$, then

$$\begin{aligned} y &= -25 - 5(0) \\ &= -25. \end{aligned}$$

The coordinates are $(0; -25)$.

Determine the x-intercept: if $y = 0$, then

$$\begin{aligned} 0 &= -25 - 5x \\ 25 &= -5x \\ -5x &= 25 \\ \frac{-5x}{-5} &= \frac{25}{-5} \\ x &= -5. \end{aligned}$$

The coordinates are $(-5; 0)$.

4.2.3 The slope of the straight line

The steepness with which straight lines ascend or descend is called the slope.

The slope, a, is the ratio of the change in y values to a given change in x values.

In terms of two arbitrary points, P_1 with coordinates $(x_1; y_1)$ and P_2 with coordinates $(x_2; y_2)$ on the straight line, we can write:

$$\text{slope} = a = \frac{y_2 - y_1}{x_2 - x_1}.$$

This is depicted in figure 4.2.4.

Now it is clear from figure 4.2.4 that a is a measure of the steepness of a straight line. The greater the change in y for a given change in x, the steeper the line. Furthermore, as we saw in figure 4.2.3, if $a > 0$, then the line is ascending (from left to right); if $a < 0$, then the line is descending.

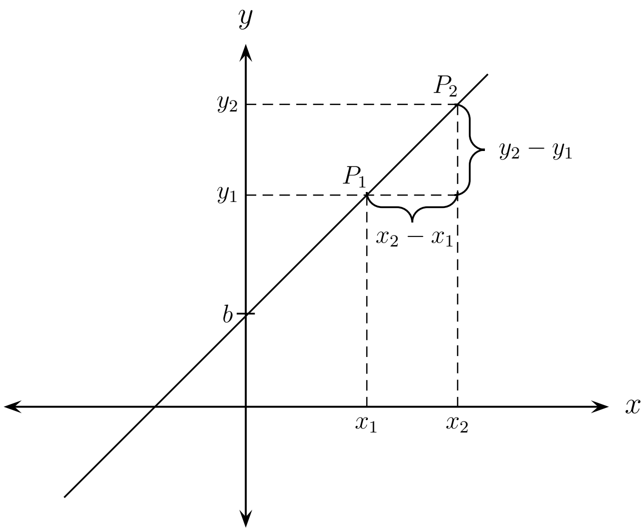


Figure 4.2.4

4.2.4 Using two points to determine the equation of a straight line

The expressions of the previous paragraphs can be used to determine the specific functional expression for the straight line passing through two given points, as the next example shows.

Determine the expression for the straight line passing through the points $(1; 3)$ and $(3; 7)$.

The general expression is

$$y = ax + b.$$

But

$$a = \frac{y_2 - y_1}{x_2 - x_1}$$

for any two points. Taking $(x_1; y_1) = (1; 3)$ and $(x_2; y_2) = (3; 7)$, we find

$$\begin{aligned} a &= \frac{7-3}{3-1} \\ &= \frac{4}{2} \\ &= 2. \end{aligned}$$

The general expression thus reduces to

$$y = 2x + b.$$

How do we find the value of b? Since the line must pass through both given points, either point can be used. Substitute the x and y values of the first point in $y = 2x + b$.

This gives

$$\begin{aligned} 3 &= 2 \times 1 + b \\ 3 &= 2 + b \\ 2 + b &= 3 \\ b &= 3 - 2 \\ &= 1. \end{aligned}$$

The expression for the line passing through the two points $(1;3)$ and $(3;7)$ is therefore

$$y = 2x + 1.$$

Note:

1. We could just as well have used the point $(3;7)$ to find the value of b, namely

$$\begin{aligned} 7 &= 2 \times 3 + b \\ 7 &= 6 + b. \end{aligned}$$

   Subtracting 6 from both sides gives

$$b = 1.$$

2. It does not matter which point we call P_1 and which P_2. Had we numbered them the other way around above we would have found:

$$\begin{aligned} a &= \frac{3-7}{1-3} \\ &= \frac{-4}{-2} \\ &= 2 \end{aligned}$$

   as before.

Activity

Determine the expression for the straight line passing through the points $(-2;\ 8)$ and $(4;\ 1)$.

Answer

The general expression is

$$y = ax + b$$

with

$$a = \frac{y_2 - y_1}{x_2 - x_1}$$

for any two points. Taking $(x_1;\ y_1) = (-2;\ 8)$ and $(x_2;\ y_2) = (4;\ 1)$, we find

$$\begin{aligned} a &= \frac{1-8}{4-(-2)} \\ &= \frac{-7}{6}. \end{aligned}$$

The general expression thus reduces to

$$y = \frac{-7}{6}x + b.$$

How do we find the value of b? Since the line must pass through both given points, either point can be used. Substitute the x- and y-value of the first point in $y = \frac{-7}{6}x + b$. This gives

$$\begin{aligned} 8 &= \frac{-7}{6} \times -2 + b \\ 8 &= \frac{7}{3} + b. \end{aligned}$$

Subtract $\frac{7}{3}$ from both sides to find that

$$b = \frac{17}{3}.$$

The expression for the line passing through the two points $(-2;\, 8)$ and $(4;\, 1)$ is therefore

$$y = \frac{-7}{6}x + \frac{17}{3}.$$

Sometimes the two points are not given to you, but you must unravel them from the information given, as illustrated in the following example.

Miriam bakes vetkoek. If she bakes 20 vetkoek, her cost is R8,00, and if she bakes 40 vetkoek, her cost is R13,00. Determine the linear cost function if it is assumed that a linear relationship exists between the cost and the number of vetkoek baked.

Let x represent the number of vetkoek baked.

Let y represent the cost of baking the vetkoek.

The following data for x and y are given:

x	y
20	8
40	13.

Thus, two data points that satisfy the linear relationship are

$$(20;\, 8) \quad \text{and} \quad (40;\, 13).$$

The general expression is

$$y = ax + b,$$

with

$$a = \frac{y_2 - y_1}{x_2 - x_1}$$

for any two points. Taking $(x_1;\, y_1) = (20;\, 8)$ and $(x_2;\, y_2) = (40;\, 13)$, we find

$$\begin{aligned} a &= \frac{13 - 8}{40 - 20} \\ &= \frac{5}{20} \\ &= \frac{1}{4} \\ &= 0{,}25. \end{aligned}$$

The general expression thus reduces to

$$y = 0{,}25x + b.$$

How do we find the value of b? Since the line must pass through both given points, either point can be used. Substitute the x- and y-value of the first point in $y = 0{,}25x + b$. This gives

$$\begin{aligned} 8 &= 0{,}25 \times 20 + b \\ 8 &= 5 + b. \end{aligned}$$

Subtracting 5 from both sides gives

$$b = 3.$$

The expression for the line passing through the two points $(20; 8)$ and $(40; 13)$ is therefore

$$y = 0{,}25x + 3.$$

Activity

Mr BR Wash sells BB (Brighter and Better) washing powder. If he charges R19 per box, he has a weekly demand of 26 000 boxes and if he charges R21 per box the weekly demand is 16 000. If p is the price per box and d is the weekly demand, derive an expression for the linear weekly demand.

Answer

Let d represent the weekly demand and p the price per box.

Thus, the weekly demand (d) can be written as a linear function in terms of the price (p).

The general expression is

$$d = ap + b.$$

But

$$a = \frac{d_2 - d_1}{p_2 - p_1}$$

for any two points. Taking $(p_1; d_1) = (19; 26\,000)$ and $(p_2; d_2) = (21; 16\,000)$, we find

$$\begin{aligned} a &= \frac{16\,000 - 26\,000}{21 - 19} \\ &= \frac{-10\,000}{2} \\ &= -5\,000. \end{aligned}$$

The general expression thus reduces to

$$d = -5\,000p + b.$$

How do we find the value of b? Since the line must pass through both given points, either point can be used. Substitute the p- and d-value of the first point in $d = -5\,000p + b$:

$$\begin{aligned} 26\,000 &= -5\,000 \times 19 + b \\ 26\,000 &= -95\,000 + b. \end{aligned}$$

Add 95 000 to both sides:

$$b = 121\,000.$$

The expression for the line passing through the two points (19; 26 000) and (21; 16 000) is therefore

$$d = -5\,000p + 121\,000.$$

4.2.5 Representing a function on a set of axes

Only two points are needed to determine an equation of a straight line. If you are not convinced of this, just mark two points on a piece of paper and try to put more than one straight line through them. To draw the graph of a straight line we make use of the general method:

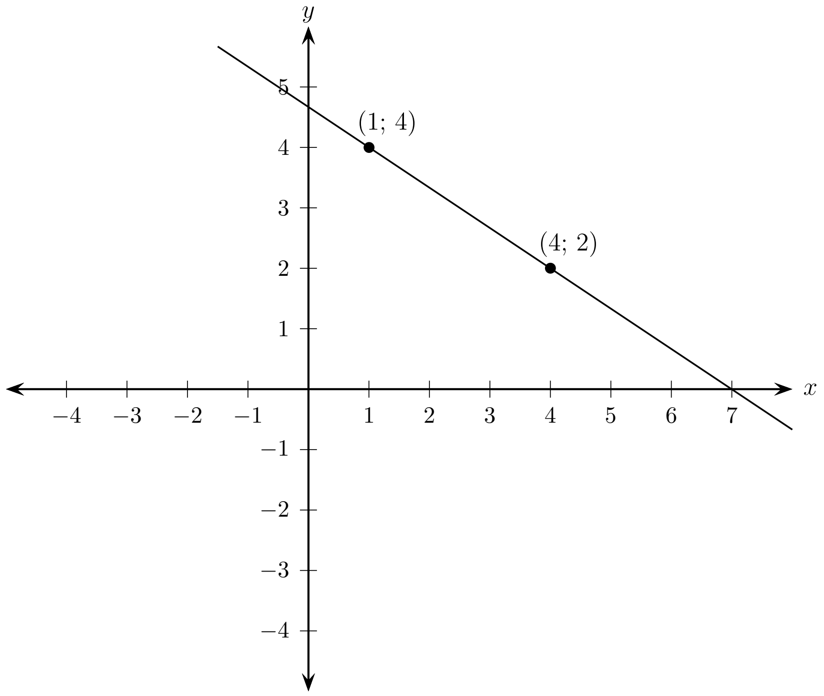
1. Draw your axes and label them.
2. Choose the scale of the axes.
3. Plot the two given points.
4. Draw a line through the two plotted points.

Activity

Plot the straight line that passes through the points $(1; 4)$ and $(4; 2)$.

Answer

The graph is given below:



What do we do if we do not have two points, but only the expression of the line $y = ax + b$?

In order to draw a straight line we only need two points. These two points must, however, consist of a x-value and a corresponding y-value. To obtain these points we substitute any two values of the one variable into the function and calculate the other. Two useful points to use, and which are easy to calculate, are the y-intercept (where $x = 0$) and the x-intercept (where $y = 0$). Thus, select $x = 0$ and calculate y, and then select $y = 0$ and calculate x.

I should point out that there is no reason why we must choose the points where the line crosses the axes. Any two points which satisfy the expression $y = ax + b$ will do. In fact, we are often only interested in the first quadrant, that is, points for which both x and y are greater than or equal to zero. In many actual problems, such as the production process referred to above, the variables can only assume positive values. In such cases we can select any two points in the first quadrant to draw the straight line.

Draw the line

$$y = 2 + x.$$

To draw the line we need to determine two points through which the line passes.

If $x = 0$ then

$$\begin{aligned} y &= 2 + 0 \\ &= 2. \end{aligned}$$

The point is (0; 2).

If $y = 0$ then

$$\begin{aligned} 0 &= 2 + x \\ 2 + x &= 0 \\ x &= -2. \end{aligned}$$

The point is (−2; 0).

Plot the two points and draw a line through the two points.

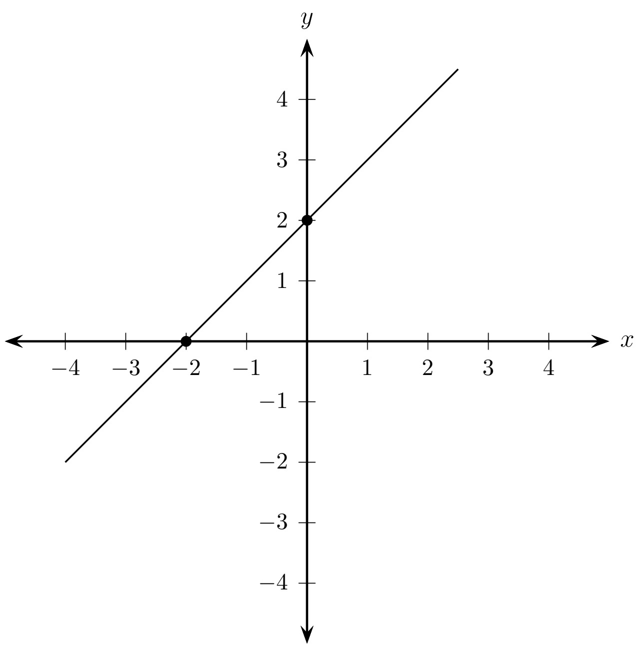


Figure 4.2.5

Activity

Draw the following straight lines:

1. $y = x + 1$
2. $y = 2 - 4x$
3. $y = 9x - 6$
4. $y = -25 - 5x$

Answer

1. If $x = 0$, then

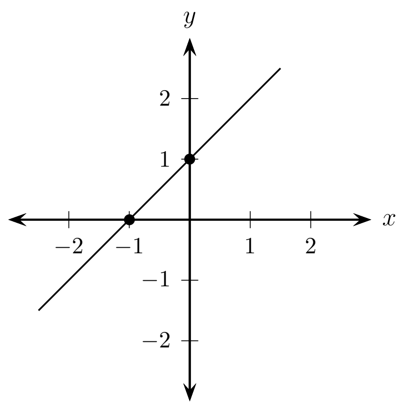
$$\begin{aligned} y &= 0+1 \\ &= 1. \end{aligned}$$

If $y = 0$, then

$$\begin{aligned} 0 &= x+1 \\ x+1 &= 0 \\ x &= -1. \end{aligned}$$

The two points on the line are (0; 1) and (−1; 0).

The graph is given below.



2. If $x = 0$, then

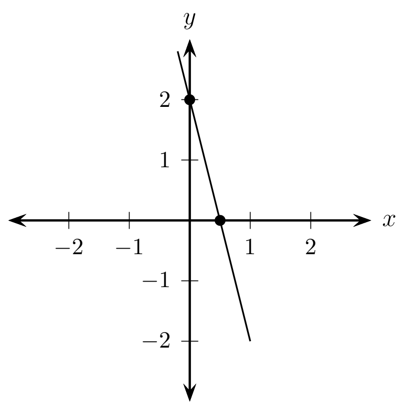
$$\begin{aligned} y &= 2-4(0) \\ &= 2. \end{aligned}$$

If $y = 0$, then

$$\begin{aligned} 0 &= 2-4x \\ 4x &= 2 \\ x &= 0{,}5. \end{aligned}$$

The two points on the line are (0; 2) and (0,5; 0).

The graph is given below.



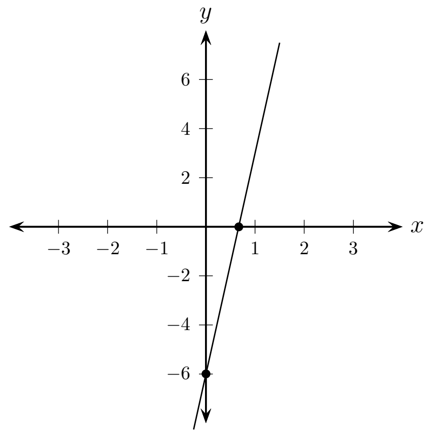
3. If $x = 0$, then

$$\begin{aligned} y &= 9(0) - 6 \\ &= -6. \end{aligned}$$

If $y = 0$, then

$$\begin{aligned} 0 &= 9x - 6 \\ 6 &= 9x \\ \frac{9x}{9} &= \frac{6}{9} \\ x &= \frac{2}{3}. \end{aligned}$$

The two points on the line are $(0; -6)$ and $\left(\frac{2}{3}; 0\right)$. The graph is given below.



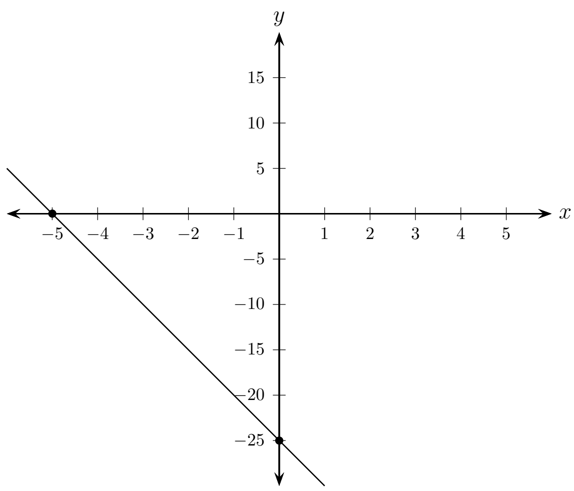
4. If $x = 0$, then

$$\begin{aligned} y &= -25 - 5(0) \\ &= -25. \end{aligned}$$

If $y = 0$, then

$$\begin{aligned} 0 &= -25 - 5x \\ 5x &= -25 \\ x &= -5. \end{aligned}$$

The two points on the line are $(0; -25)$ and $(-5; 0)$. The graph is given below.



4.2.6 Special cases

- **The case for which the constant term is zero, this is $b = 0$.**

  This means that the intercepts on both axes are zero. That is, the line goes through the origin as shown in figure 4.2.6.

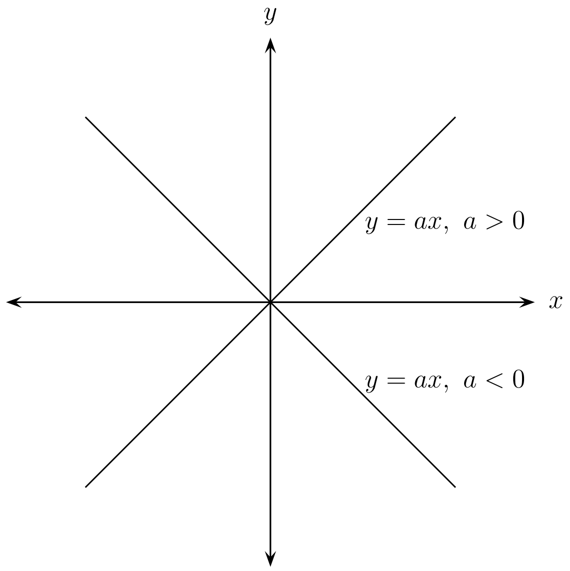


Figure 4.2.6

- **The case of a zero valued slope, this is $a = 0$.**

  What does this mean? Looking at our expression for a, that is

$$\frac{y_2 - y_1}{x_2 - x_1},$$

we see that this can only be the case if $y_2 = y_1$. That is, if the function values are the same and y is not dependent on x, it is in fact a constant. This is represented by a straight line parallel to the x-axis (a horizontal line) as depicted in figure 4.2.7.

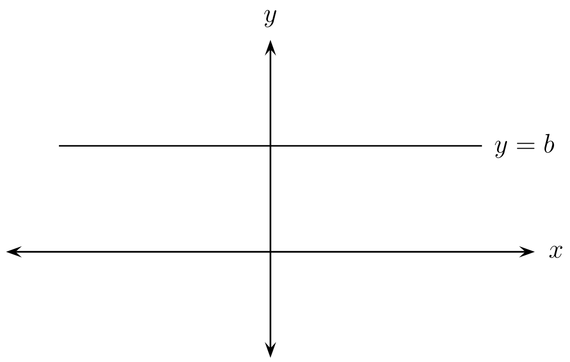


Figure 4.2.7

Notice that in this case there is no intercept on the x-axis.

- **A straight line parallel to the y-axis.**

  In this case we would have $x_2 = x_1$ and

  $$\frac{y_2 - y_1}{0}.$$

  Division by zero is not defined. We say that the slope becomes **infinite** in this case. The line is **vertical**, that is, as shown in figure 4.2.8.

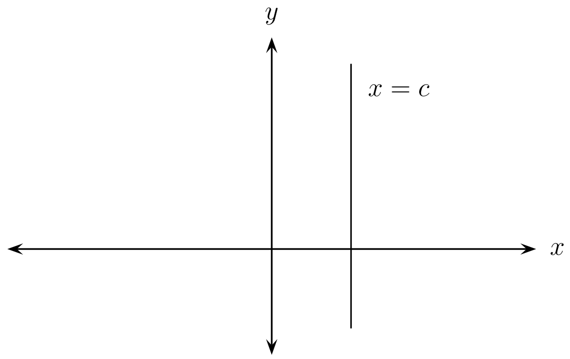


Figure 4.2.8

  As indicated, the expression for this line is $x = c$, where c is the intercept on the x-axis. There is no intercept on the y-axis.

- **Two straight lines y and $\overline{y}$, which have the same slope but different y-intercepts.**

  For example:

  $$y = ax + b \quad \text{and} \quad \overline{y} = ax + \overline{b}$$

  have the same slope a, but different intercepts on the y-axis, namely b and $\overline{b}$. (Note the use of the bar on y and b to distinguish the two cases.) If we subtract the one expression from the other we obtain

  $$y - \overline{y} = b - \overline{b}.$$

  This indicates that the distance between the two lines does not depend on the value of x, it is constant. In other words, the lines are parallel. This is depicted in figure 4.2.9.

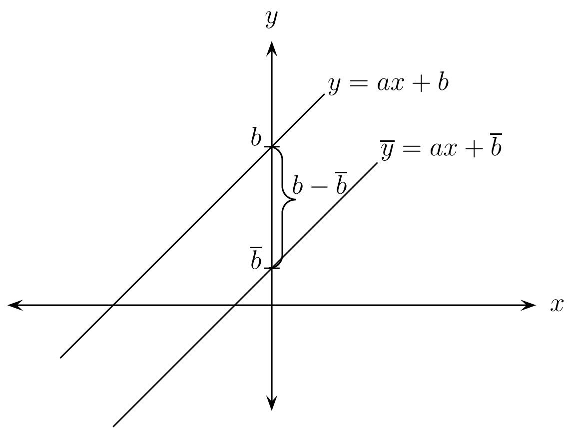


Figure 4.2.9

Activity

Draw the following lines on the same set of axes:

1. $y = 4$
2. $x = 6$
3. $y = 2x$
4. $y = 2x + 4$
5. $y = 2x - 1$

Can you notice anything in regard to 3, 4 and 5?

Answer

1. This is a horizontal line at $y = 4$.
2. This is a vertical line at $x = 6$.
3. If $x = 0$, then

$$\begin{aligned} y &= 2(0) \\ &= 0. \end{aligned}$$

The point is (0; 0).

If $y = 0$, then

$$\begin{aligned} 0 &= 2x \\ x &= 0. \end{aligned}$$

The point is (0; 0).

Thus, we still have only one point. Take any other y-value and determine a x-value.

If $y = 1$, then

$$\begin{aligned} 1 &= 2x \\ x &= \frac{1}{2}. \end{aligned}$$

Thus two points on the line are (0; 0) and $(\frac{1}{2}; 1)$

4. If $x = 0$, then

$$\begin{aligned} y &= 2(0) + 4 \\ &= 4. \end{aligned}$$

If $y = 0$, then

$$\begin{aligned} 0 &= 2x + 4 \\ x &= -2. \end{aligned}$$

Thus, two points on the line are (0; 4) and (−2; 0).

5. If $x = 0$, then

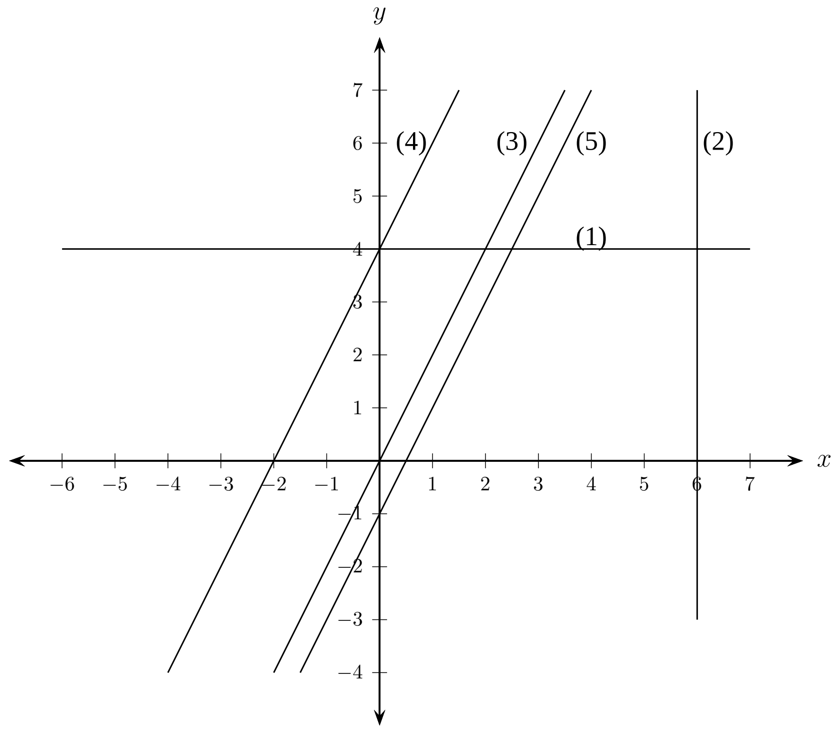
$$\begin{aligned} y &= 2(0) - 1 \\ &= -1. \end{aligned}$$

If $y = 0$, then

$$\begin{aligned} 0 &= 2x - 1 \\ x &= \frac{1}{2}. \end{aligned}$$

Thus, two points on the line are $(0; -1)$ and $(\frac{1}{2}; 0)$.

Plot the data points for the different graphs and draw the necessary lines through them. The following graph is obtained.



Lines (3), (4) and (5) are parallel. The three lines have the same slope.

Summary

The general expression for a straight line or linear function is

$$y = ax + b$$

where ***b*** is the **intercept** on the ***y*-axis**, and ***a*** is the **slope** of the line.

The formula for the slope a in terms of the coordinates $(x_1; y_1)$ and $(x_2; y_2)$ of two points on the line is

$$a = \frac{y_2 - y_1}{x_2 - x_1}.$$

Exercise 4.1

1. (a) Determine the equation of the straight line through the points (1; 2) and (3; 3).
   (b) Find the intercepts on the x- and the y-axe of the line in (a).
   (c) Is the line in (a) parallel to the line
   $$y = 2 + x?$$
   Why or why not?
   (d) Draw the lines of (a) and (c) on one graph.
2. Consider the lines
   $$y = 5 + 2x$$
   and
   $$y = 2 + x.$$
   What are their intercepts on the axes? Are they parallel or not? What is the vertical distance between the lines at
   $$x = 3{,}5?$$
3. Draw the following lines on one graph:
   (a) $x = 2$
   (b) $y = 4x$
   (c) $y = -2x - 3$
4. A bus agency has room for 60 people on a bus tour. If they charge R6 000 per person, they will be able to fill the bus. They know from experience that if they increase the price of the tour by R500 they will lose three customers. Determine the price function if the price p (in rand) is a linear function of the demand (number of customers).

Study unit 4.3 Quadratic functions

Learning objectives: *On completion of this study unit you should be able to*

- *explain the different characteristics of a quadratic function*
- *determine the roots and the vertex of a quadratic function*
- *graphically represent a quadratic function*

A quadratic function is a function where the relationship between the independent variable x and the dependent variable y have the form

$$y = f(x) = ax^2 + bx + c$$

where a, b and c are constants, and $a \neq 0$ ($a \neq 0$ means that a may not be zero and is read "a not equal to 0").

What are the important properties of the graph of a quadratic function, or **parabola** as it is called?

4.3.1 Shape

The two basic shapes of quadratic functions are illustrated in figure 4.3.1 below.

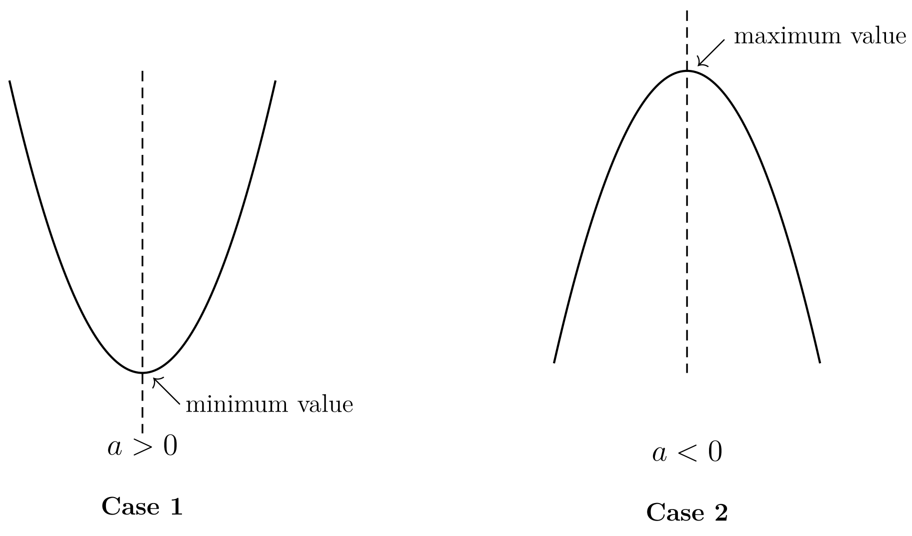


Figure 4.3.1

In the first case there is a **least *y*-value** at which the curve turns – the function has a **minimum value**. In the second case there is a **greatest *y*-value** at which the curve turns. The function has a **maximum value**.

How do we know which one of the two forms we are dealing with without actually sketching the curve? The answer lies in the value of the coefficient of x^2, that is, the constant a.

If $\boldsymbol{a > 0}$ then we have a function with a **minimum value** (ie case (1) above).

If $\boldsymbol{a < 0}$ then we have a function with a **maximum value** (ie case (2) above).

Furthermore, the quadratic function is symmetric with respect to a vertical line called the axis of symmetry – shown dotted in figure 4.3.1 above. The line also passes through the lowest point or the highest point of the parabola. This point of intersection is called the **vertex** of the parabola.

4.3.2 Turning point – vertex

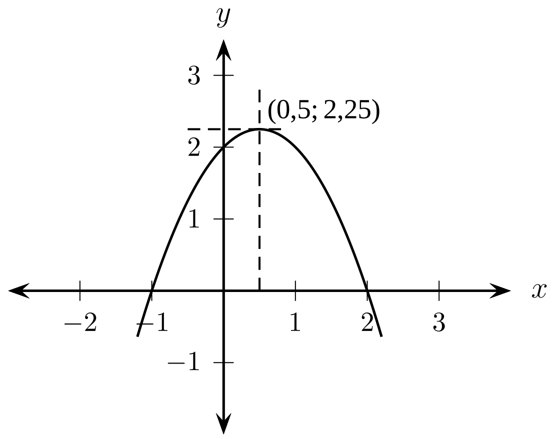
The value of x at the vertex is calculated by using the formula

$$x_m = -\frac{b}{2a}.$$

The **value of the function at the vertex**, which is the minimum or maximum value of the function, is found by substituting this value of x into the quadratic function.

Activity

Consider the graph of the following parabola: $y = f(x) = -x^2 + x + 2$.



1. Determine the value of x at the vertex.
2. Determine the value of y at the vertex.

Answer

From $y = -x^2 + x + 2$, we have that

$$a = -1,\ b = 1 \text{ and } c = 2.$$

1. The value of x at the vertex is

$$\begin{aligned} x_m &= \frac{-b}{2a} \\ &= \frac{-(1)}{2(-1)} \\ &= \frac{-1}{-2} \\ &= \frac{1}{2} \\ &= 0{,}5. \end{aligned}$$

2. The value of y at the vertex is

$$\begin{aligned} y &= f\left(\frac{-b}{2a}\right) \\ &= f(0{,}5) \\ &= -1(0{,}5)^2 + 1(0{,}5) + 2 \\ &= -0{,}25 + 0{,}5 + 2 \\ &= 2{,}25. \end{aligned}$$

These values correspond with those on the graph.

4.3.3 Intercepts on the axes

The **intercept on the y-axis** is easily found since it is simply the value of the function for $x = 0$, namely $f(0)$. Thus the intercept on the y-axis is given by

$$f(0) = c.$$

The **intercepts on the x-axis** are a little more difficult to find. As you can see there are two intercepts on the x-axis. These are the values of x for which the function is zero:

$$ax^2 + bx + c = 0.$$

The values of the two intercepts are given by

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}.$$

The quantity

$$b^2 - 4ac$$

in the above expressions is known as the **discriminant** since it discriminates between different types of intercepts.

1. If $b^2 - 4ac > 0$,

   then we can evaluate its square root and determine the two separate intercepts using the above expressions. Note that the intercepts are equally far from the x value which determines the vertex. In fact they are often written in the form

   $$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

   to emphasise the fact.

2. If $b^2 - 4ac = 0$,

   both expressions reduce to the same, namely

   $$x = \frac{-b}{2a}$$

   which is also the value of x at the vertex. In other words, in this case the vertex of the parabola just touches the x-axis at this point.

3. If $b^2 - 4ac < 0$,

   we cannot evaluate the square root since the square root of a negative number is not a real number. In this case the parabola has no intercept on the x-axis, nor does it touch it. The properties mentioned above are illustrated in figure 4.3.2. In the graphs

   $$d = b^2 - 4ac.$$

Take note that the curves are all symmetrical about a vertical line through the vertex. This means that there are always two x-values which give the same y-value (the vertex excluded).

The exact position of the intercepts, that is, whether they are to the left or right of the y-axis, will also depend on whether b is positive or negative. Also note that $a = 0$ is not allowed, since the x^2 term then falls away and the quadratic reduces to a linear function. Try it! The three specific cases which can occur are illustrated in figure 4.3.2.

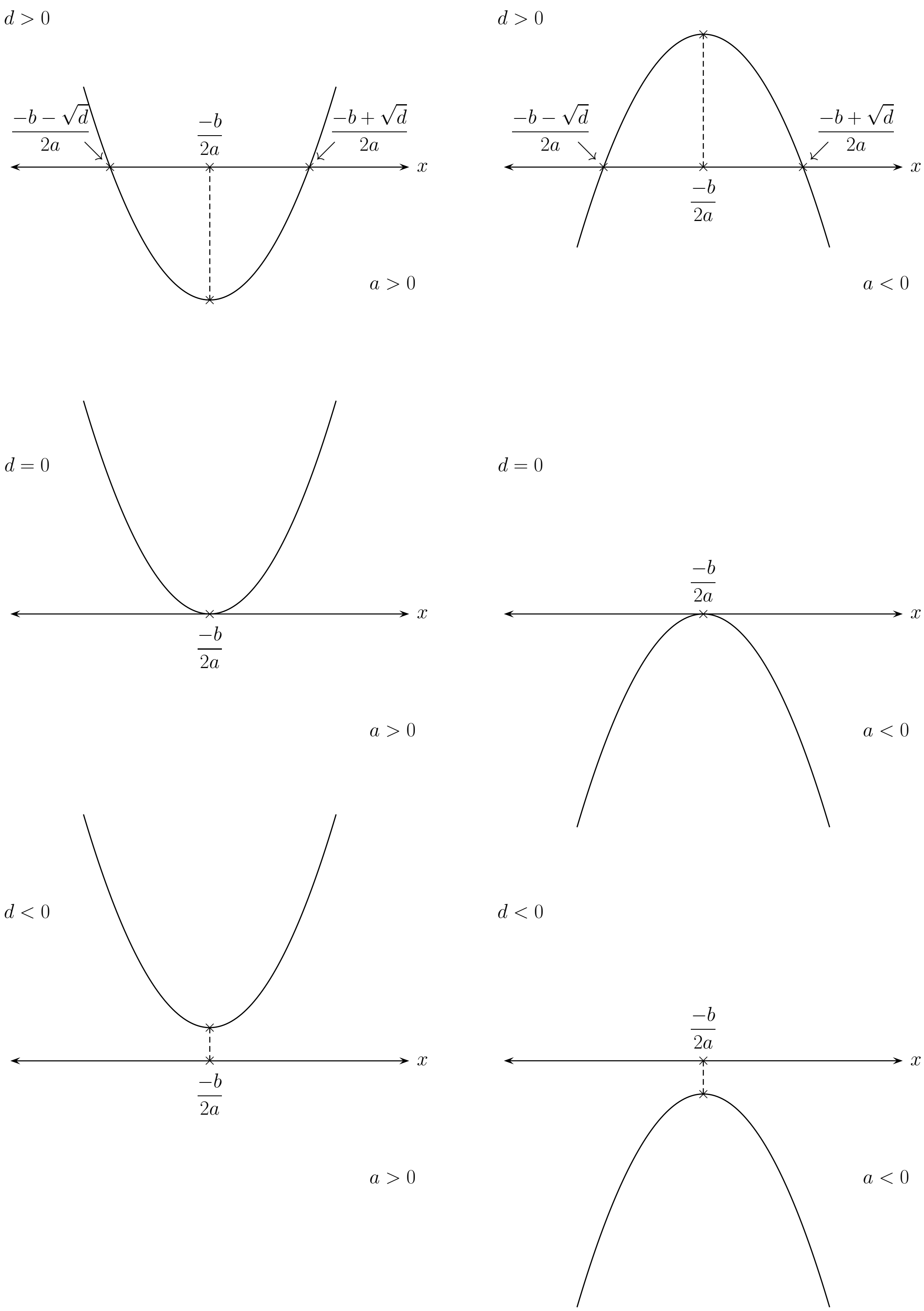

$d > 0$
$\frac{-b-\sqrt{d}}{2a}$
$\frac{-b}{2a}$
$\frac{-b+\sqrt{d}}{2a}$
x
$a > 0$
$d > 0$
$\frac{-b-\sqrt{d}}{2a}$
$\frac{-b+\sqrt{d}}{2a}$
x
$\frac{-b}{2a}$
$a < 0$
$d = 0$
x
$\frac{-b}{2a}$
$a > 0$
$d = 0$
$\frac{-b}{2a}$
x
$a < 0$
$d < 0$
x
$\frac{-b}{2a}$
$a > 0$
$d < 0$
$\frac{-b}{2a}$
x
$a < 0$

Figure 4.3.2

Activity

Consider the quadratic function

$$y = -2x^2 + 8x - 6.$$

Determine the value of x at the vertex and the intercepts on the axes.

Answer

In this case,

$$a = -2, \ \ b = 8 \ \text{ and } \ c = -6.$$

The value of x at the vertex is

$$\begin{aligned} x &= -\frac{b}{2a} \\ &= -\frac{8}{2 \times (-2)} \\ &= \frac{8}{4} \\ &= 2. \end{aligned}$$

Since $a < 0$, the function has a maximum. The value of the function at the maximum is

$$\begin{aligned} f(2) &= -2 \times 2^2 + 8 \times 2 - 6 \\ &= 2. \end{aligned}$$

The discriminant is

$$\begin{aligned} b^2 - 4ac &= 8^2 - 4 \times (-2)(-6) \\ &= 64 - 48 \\ &= 16. \end{aligned}$$

Since this is greater than 0, there are two intercepts on the x-axis which are given by

$$\begin{aligned} x &= \frac{-b - \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-8 - \sqrt{16}}{2 \times (-2)} \\ &= \frac{-8 - 4}{-4} \\ &= \frac{-12}{-4} \\ &= 3 \end{aligned}$$

and

$$\begin{aligned} x &= \frac{-b + \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-8 + \sqrt{16}}{2 \times (-2)} \\ &= \frac{-8 + 4}{-4} \\ &= \frac{-4}{-4} \\ &= 1. \end{aligned}$$

4.3.4 Slope

One final point. You may have noted that I have not referred to the slope of a quadratic function. Looking at the graphs the quadratic function obviously has a slope, but it is constantly changing, unlike the linear function which has a constant slope. That is, the slope of a quadratic function is never the same for any two points on the graph.

This is illustrated in figure 4.3.3 below with the use of **tangents**. A tangent is a straight line that just touches the graph at one point as illustrated.

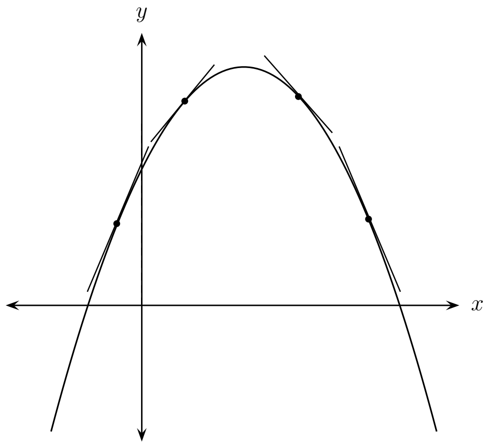


Figure 4.3.3

You will not need to calculate the slope of a non-linear function in this course and the main point that I want to emphasise is, as I have shown, that it is ever-changing, in contrast to the straight line where it is a constant.

4.3.5 Drawing a quadratic function

Note that when asked to draw a quadratic function it is not necessary to plot it in great detail. Just indicate the intercepts and the vertex and the approximate shape. Now try a few cases by yourself.

Activity

Determine the intercepts on the axes and the vertices of the following quadratic functions and draw their graphs:

1. $y = 2x^2 - x - 3$
2. $y = 4x^2 - 16x + 16$
3. $y = -3x^2 + 3x - 2$

Answer

1. From

$$y = 2x^2 - x - 3$$

we have that

$$a = 2,\ b = -1 \text{ and } c = -3.$$

Since $a > 0$ the function has a minimum.

The value of x at the vertex is

$$\begin{aligned} x_m &= \frac{-b}{2a} \\ &= -\frac{-1}{2\times 2} \\ &= \frac{1}{4}. \end{aligned}$$

The value of the function at the vertex, which is the minimum, is

$$\begin{aligned} y &= f\left(\frac{1}{4}\right) \\ &= 2\left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right) - 3 \\ &= -3\frac{1}{8}. \end{aligned}$$

The intercept on the y-axis is

$$c = -3.$$

Before calculating the intercepts on the x-axis, we first determine the value of the discriminant. The discriminant is

$$\begin{aligned} b^2 - 4ac &= (-1)^2 - 4\times 2\times(-3) \\ &= 1 + 24 \\ &= 25. \end{aligned}$$

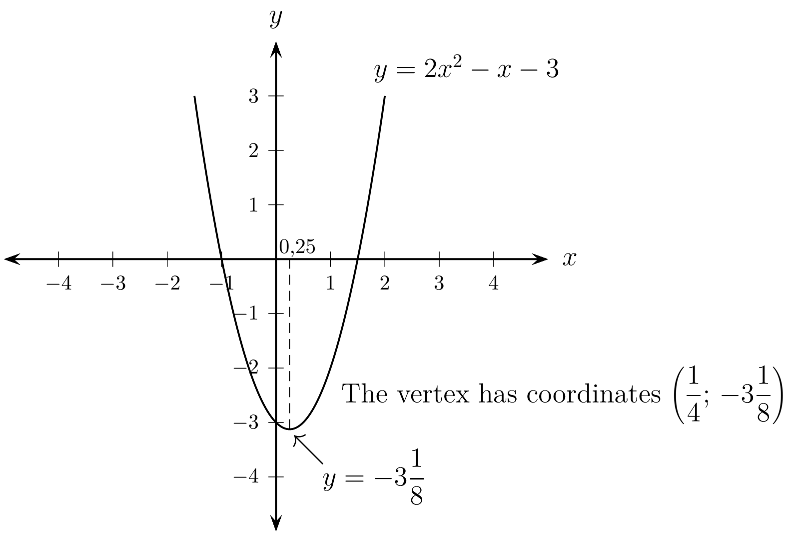
Since this is greater than 0, two intercepts exist. Thus the intercepts are

$$\begin{aligned} x &= \frac{-b - \sqrt{b^2-4ac}}{2a} \\ &= \frac{-(-1) - \sqrt{25}}{2\times 2} \\ &= \frac{1-5}{4} \\ &= \frac{-4}{4} \\ &= -1 \end{aligned}$$

and

$$\begin{aligned} x &= \frac{-b + \sqrt{b^2-4ac}}{2a} \\ &= \frac{-(-1) + \sqrt{25}}{2\times 2} \\ &= \frac{1+5}{4} \\ &= \frac{6}{4} \\ &= 1\frac{1}{2}. \end{aligned}$$

The graph of the quadratic function is shown below.



2. From

$$y = 4x^2 - 16x + 16$$

we have that

$$a = 4,\ b = -16 \text{ and } c = 16.$$

Since $a > 0$ the function has a minimum.
The value of x at the vertex is

$$\begin{aligned} x_m &= \frac{-b}{2a} \\ &= -\frac{-16}{2 \times 4} \\ &= 2. \end{aligned}$$

The minimum value of the function is thus

$$\begin{aligned} y &= f(2) \\ &= 4 \times 2^2 - 16 \times 2 + 16 \\ &= 0. \end{aligned}$$

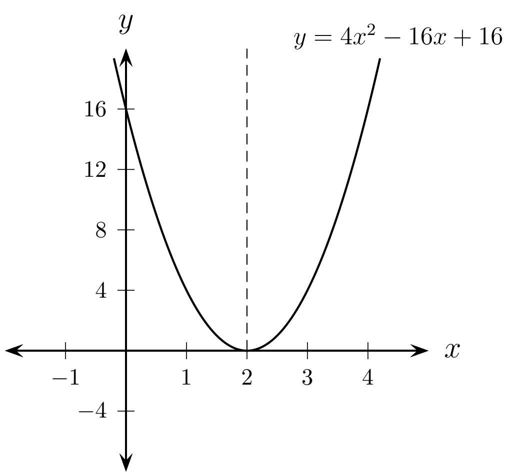
The intercept on the y-axis is

$$c = 16.$$

The discriminant is

$$\begin{aligned} b^2 - 4ac &= (-16)^2 - 4 \times 4 \times 16 \\ &= 256 - 256 \\ &= 0. \end{aligned}$$

Thus, the parabola just touches the x-axis at $x = 2$.
The graph of the quadratic function is shown below.



3. From

$$y = -3x^2 + 3x - 2$$

we have that

$$a = -3,\ b = 3 \text{ and } c = -2.$$

The value of x at the vertex is

$$\begin{aligned} x_m &= \frac{-b}{2a} \\ &= \frac{-3}{2 \times -3} \\ &= \frac{1}{2}. \end{aligned}$$

Since $a < 0$ the function has a maximum. The maximum value is

$$\begin{aligned} y &= f\left(\frac{1}{2}\right) \\ &= -3\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right) - 2 \\ &= -1\frac{1}{4}. \end{aligned}$$

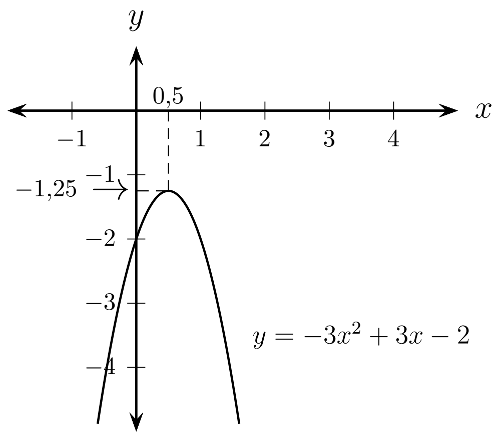
The intercept on the y-axis is at

$$c = -2.$$

The discriminant is

$$\begin{aligned} b^2 - 4ac &= 3^2 - 4 \times (-3) \times (-2) \\ &= 9 - 24 \\ &= -15. \end{aligned}$$

Since the discriminant is less than zero there are no intercepts on the x-axis. The graph of the quadratic function is shown below.



To conclude, some information on the applications of quadratic functions. I suspect that you will recognise, even if it is only an intuitive feeling that you have, that the potential uses for the quadratic function are considerable. It is evident that they can be used to model situations in which the dependent variable is expected to peak, or, alternatively, to pass through a dip, for some value of the independent variable; for example demand, supply and profit functions.

Summary

Properties of the quadratic function

$$f(x) = ax^2 + bx + c \quad \text{where} \quad a \neq 0:$$

1. If $a > 0$ the parabola opens upwards (smiling face or it holds water) and if $a < 0$ it opens downwards (sad face or does not hold water).
2. The vertex of the parabola is
$$\left(-\frac{b}{2a}; f\left(-\frac{b}{2a}\right)\right).$$
3. The y-intercept is
$$f(0) = c.$$
4. The x-intercepts (if any) are found by solving $f(x) = 0$ with
$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}.$$
5. It was pointed out that the slope of a quadratic is ever-changing in contrast to the straight line which has a constant slope.

Exercise 4.2

1. Determine the intercepts on the axes and the vertices of each of the following quadratic functions and sketch the curves:
   (a) $y = -0{,}4x^2 + 0{,}2x + 1{,}2$
   (b) $y = -x^2 - 2x - 1$
   (c) $y = x^2 + 4x + 5$
   (d) $y = -x^2 + 9$
2. If
$$d = p^2 - 45p + 520$$
describes a weekly demand for a certain ice cream in litres, with p the price per litre and d the demand, what is the price per litre that minimises the weekly demand? What is the minimum weekly demand?

Study unit 4.4 Exponential and logarithmic functions

Learning objectives: *On completion of this study unit you should know and be able to explain*

- *what an exponential function is*
- *what a logarithm to base 10 is and what a logarithm to base e is*

4.4.1 Exponential function

The exponential function is one of the most useful functions and is found in virtually every field where mathematics is applied.

The exponential function with base a is the function defined by

$$y = f(x) = a^x$$

where $a > 0$ and $a \neq 1$ and a is a real constant.

It is called the exponential function since the independent variable x appears as an exponent.

If the base $\boldsymbol{a > 1}$, then the exponential function is an **increasing function** of x, that is, as x increases from large negative values to large positive values, the function values constantly increase. Conversely, if $\boldsymbol{a < 1}$ then the exponential function is a **decreasing function** of x. In both cases the functions are positive for all values of x and they all pass through the point $(0; 1)$.

Thus, graphically it looks like the following:

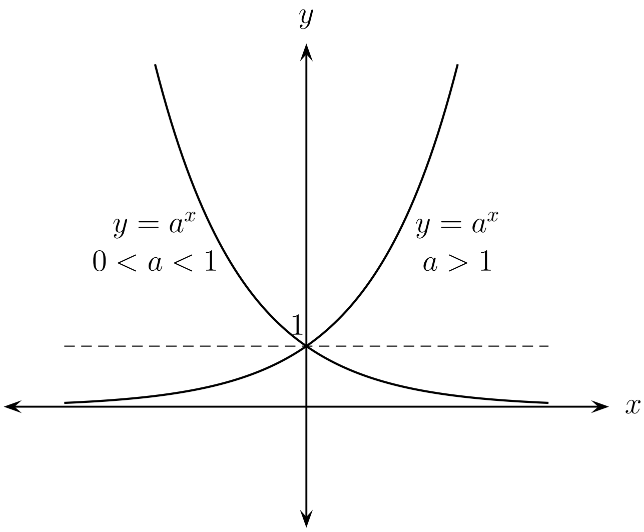


Figure 4.4.1

Exponential functions are widely used. For example exponential functions with base $a > 1$ are known as **growth curves** or functions, because of their increasing property. They are often used to describe growth in various types of population, bacterial or animal, which explains the expression "exponential growth". As you know, they are also applicable to compound interest growth.

On the other hand, exponential functions with base $a < 1$ are known as **decay functions** or curves, because of their decreasing property. They are useful for modeling certain "negative growth" phenomena, such as radioactive decay processes, price-demand curves and non-linear depreciation.

4.4.2 Logarithmic function

The logarithmic function is very closely related to the exponential function. If

$$y = a^x,$$

to what power (ie x) must a be raised to get an answer of y? We call this a logarithm. By definition, we say: if

$$y = a^x$$

with $a > 0$ and $a \neq 1$ then

$$\log_a y = x$$

which is read as "the logarithm of y to the base a is x". Note that $y > 0$.

Thus, you see that the logarithm (or log as it is often referred to) function is completely determined by the corresponding exponential function. In other words, where the exponential function allowed us to "read from x to y" as it were, the logarithmic function allowed us to "read back from y to x".

The following example illustrates this. Because

$$10^2 = 100$$

we can see that

$$\log_{10} 100 = 2.$$

In particular, two general results:
because

$$a = a^1$$

it follows that

$$\log_a a = 1$$

and because

$$a^0 = 1$$

it follows that

$$\log_a 1 = 0.$$

Activity

In each case state the meaning of

$$x = \log_a y$$

and determine x:

1. $\log_{10} 10\,000$
2. $\log_{10} 0{,}01$

Answer

1. To what power must 10 be raised to get 10 000?
   This means
   $$10\,000 = 10^x,$$
   but $10\,000 = 10^4$, thus
   $$x = 4.$$

2. To what power must 10 be raised to get 0,01?
   This means
   $$0{,}01 = 10^x,$$
   but
   $$\begin{aligned} 0{,}01 &= \frac{1}{100} \\ &= 10^{-2}, \end{aligned}$$
   thus
   $$x = -2.$$

The recommended calculator has a key for $\log_e$, that is, log to the base e, the so-called **natural logarithm**, which is usually denoted by the symbol ***ln***.

(*See Tutorial Letter 101 on how to use your calculator.*)

COMPONENT 5

Linear systems

On completion of this component you should be able to

- solve linear equations in one or two variables
- solve linear inequalities in one or two variables

CONTENTS

Study unit 5.1 Linear equations in one variable

Learning objectives: *On completion of this study unit you should be able to solve an equation in one variable algebraically.*

5.1.1 What is an equation in one variable?

An **equation in one variable** is a statement containing an = sign with algebraic expressions to the left and right of the sign, using only one variable. The values of the variable which make the statement true are called **solutions** or **roots** of the equation.

A few examples should make this definition clear. In each case the values on the right (the solutions) make the statements on the left (the equations) true:

Equation	**Solution(s) or root(s)**
$2x + 3 = x + 4$	$x = 1$
$A + 1 = 3$	$A = 2$

The process of determining which values of the variables make the statement true is known as solving the equation.

5.1.2 Solving linear equations in one variable algebraically

Linear equations are equations in which the unknown variable only appears in linear form:

$$3x + 1 = 4x + 3$$

$$5A - 40 = 0$$

$$-5s + 2 = s + 8$$

Solving linear equations is simply a matter of juggling and manipulating the equation until the variable is alone on the left-hand side. The golden rule is that we can perform the same operation on the expressions on both sides of the equal sign without altering the solution.

More specifically, we may

1. add (or subtract) the same number or expression to (or from) both sides of the equation
2. multiply (or divide) both sides of the equation by the same non-zero number or expression

(*See component 1 for a complete discussion on this.*)

The new equation obtained by any one of these operations is equivalent to the original equation.

Let us do a few examples.

1. Solve

   $$3x + 5 = 2x - 3.$$

   Subtract $2x$ from both sides:

   $$\begin{aligned} 3x - 2x + 5 &= 2x - 2x - 3 \\ x + 5 &= -3. \end{aligned}$$

Subtract 5 from both sides:

$$\begin{aligned} x+5-5 &= -3-5 \\ x &= -8. \end{aligned}$$

Note that the associative and commutative laws of addition allow us to enter the term that we are adding at any position on each side of the equation, for example whether we write

$$3x-2x+5 \quad \text{or} \quad -2x+3x+5 \quad \text{or} \quad 3x+5-2x,$$

the result is the same.

2. Solve

$$4A-25=0.$$

Add 25 to both sides:

$$\begin{aligned} 4A-25+25 &= 25 \\ 4A &= 25. \end{aligned}$$

Multiply both sides by $\frac{1}{4}$:

$$\begin{aligned} \frac{1}{4}\times 4A &= \frac{1}{4}\times 25 \\ A &= \frac{25}{4} \\ &= 6\frac{1}{4}. \end{aligned}$$

3. Solve

$$5s-6=10-\left(\frac{s}{6}\right).$$

Add $\frac{s}{6}$ to both sides:

$$\begin{aligned} 5s+\frac{s}{6}-6 &= 10-\frac{s}{6}+\frac{s}{6} \\ \left(5+\frac{1}{6}\right)s-6 &= 10 \end{aligned}$$

where we have used the distributive law to add the two terms containing the s. Then we have

$$\frac{31}{6}s-6=10 \qquad \left(5+\frac{1}{6}=\frac{30}{6}+\frac{1}{6}=\frac{31}{6}\right).$$

Add 6 to both sides:

$$\begin{aligned} \frac{31}{6}s-6+6 &= 10+6 \\ \frac{31}{6}s &= 16. \end{aligned}$$

Multiply both sides by $\frac{6}{31}$:

$$\begin{aligned} \frac{6}{31}\times\frac{31}{6}s &= 16\times\frac{6}{31} \\ s &= \frac{16\times 6}{31} \\ &= \frac{96}{31}. \end{aligned}$$

Note that if you are not adept in adding fractions together, such as $5 + \frac{1}{6}$, you may use your calculator to obtain 5,166 ... (and finally, $s = 3{,}096774194$ for the solution). (Try it!)

Activity

Solve the following:

1. $25 + 3x = 50 - 7x$
2. $y - 3 = 2y + 4$

Answer

1. The equation is

$$25 + 3x = 50 - 7x.$$

Add $+7x$ to both sides:

$$\begin{aligned} 25 + 3x + 7x &= 50 - 7x + 7x \\ 25 + 10x &= 50. \end{aligned}$$

Subtract 25 from both sides:

$$\begin{aligned} -25 + 25 + 10x &= 50 - 25 \\ 10x &= 25. \end{aligned}$$

Divide both sides by 10:

$$\begin{aligned} x &= \frac{25}{10} \\ &= 2{,}5. \end{aligned}$$

2. The equation is

$$y - 3 = 2y + 4.$$

Subtract $2y$ from both sides:

$$\begin{aligned} -2y + y - 3 &= -2y + 2y + 4 \\ -y - 3 &= 4. \end{aligned}$$

Add 3 to both sides:

$$\begin{aligned} -y - 3 + 3 &= 4 + 3 \\ -y &= 7. \end{aligned}$$

Multiply both sides by -1:

$$\begin{aligned} -1 \times -y &= -1 \times 7 \\ y &= -7. \end{aligned}$$

Although we have been working step by step here, there is no reason why you cannot use shortcuts and add several terms at once, as the next example illustrates.

Solve

$$4x + 30 = 16x - 54.$$

Add $-16x - 30$ to both sides:

$$\begin{aligned} -16x + 4x + 30 - 30 &= -16x + 16x - 54 - 30 \\ -12x &= -84. \end{aligned}$$

Multiply both sides by $-\dfrac{1}{12}$:

$$\begin{aligned} -\frac{1}{12} \times -12x &= -\frac{1}{12} \times -84 \\ x &= \frac{84}{12} \\ &= 7. \end{aligned}$$

However, if you are at all unsure, rather be on the safe side and work step by step. Any linear equation can be solved by using the following method:

1. Manipulate the linear equation by operations of the above type until it is in the form
$$ax + b = 0.$$
The left-hand side is the general expression for a linear function, hence the name linear equation.

2. This equation is solved by adding $-b$ to both sides to get
$$ax + b - b = -b \quad \text{or} \quad ax = -b.$$

3. Divide by a, assuming $a \neq 0$, to obtain the root or solution
$$x = \frac{-b}{a}.$$

Do you recognise the last expression? You should. It is, in fact, the expression for the intercept on the x-axis of the linear function $y = ax + b$. Since this intercept occurs at $y = 0$ this result should not surprise you.

In other words, we can always interpret the root of a linear equation in one variable as the x-axis intercept of the corresponding linear function as shown in figure 5.1.1.

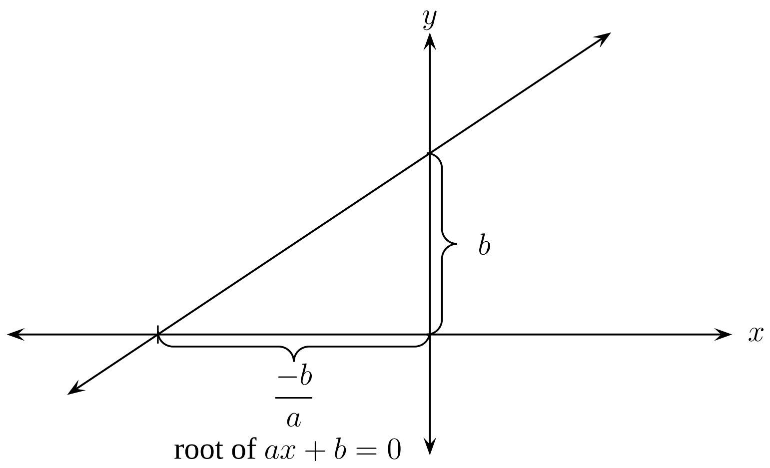


Figure 5.1.1

To summarise:

We can solve any linear equation quite easily by rearranging it into the form

$$ax + b = 0,$$

whereafter the solution is simply

$$x = \frac{-b}{a}.$$

Activity

Solve the following by writing it in the form $ax + b = 0$:

1. $5x + 6 = -3x - 10$
2. $1{,}1x + 3{,}4 = 2{,}5x + 1{,}3$

Answer

1. The equation is
$$5x + 6 = -3x - 10.$$
Add $3x + 10$ to both sides:
$$\begin{aligned} 5x + 3x + 6 + 10 &= 0 \\ 8x + 16 &= 0 \\ x &= \frac{-16}{8} \\ &= -2. \end{aligned}$$

2. The equation is
$$1{,}1x + 3{,}4 = 2{,}5x + 1{,}3.$$
Add $-(2{,}5x + 1{,}3)$ to both sides:
$$\begin{aligned} 1{,}1x - 2{,}5x + 3{,}4 - 1{,}3 &= 2{,}5x - 2{,}5x + 1{,}3 - 1{,}3 \\ (1{,}1 - 2{,}5)x + 3{,}4 - 1{,}3 &= 0 \\ -1{,}4x + 2{,}1 &= 0 \\ x &= \frac{-2{,}1}{-1{,}4} \\ &= 1{,}5. \end{aligned}$$

Study unit 5.2 Simultaneous linear equations in two variables

Learning objectives: *On completion of this study unit you should be able to solve algebraically simultaneous linear equations in two variables.*

Consider the following two functions:

$$y = 3 - 2x \text{ and } y = 2 + x.$$

If we consider them separately, then there is a whole range of y-values for the x-values.

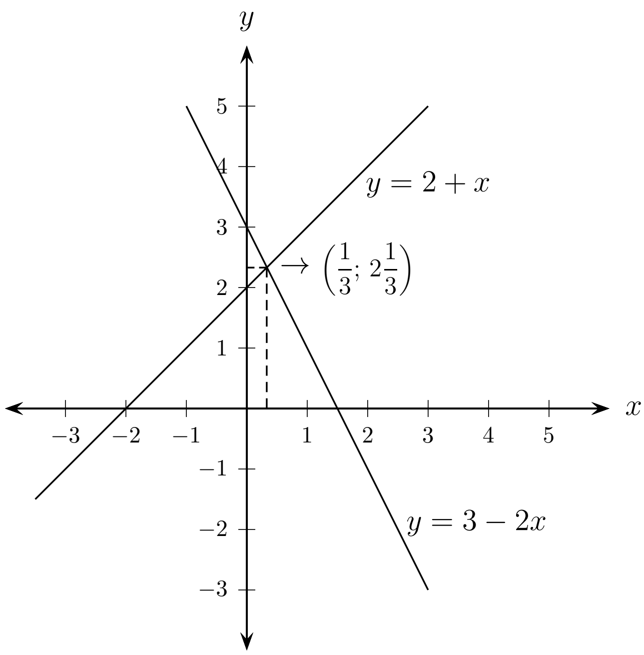


Figure 5.2.1

The point in figure 5.2.1 where the two graphs intersect is common to both functions. In this case it is the point

$$\left(\frac{1}{3}; 2\frac{1}{3}\right).$$

But how do we solve it algebraically? Set $y = 3 - 2x$ equal to $y = 2 + x$:

$$\begin{aligned} 3 - 2x &= 2 + x \\ 3 - 2 &= 2x + x \\ 1 &= 3x \\ x &= \frac{1}{3}. \end{aligned}$$

To determine the value of y we substitute $x = \frac{1}{3}$ in any one of the two equations. Substitute it in $y = 3 - 2x$, that is,

$$\begin{aligned} y &= 3 - 2 \times \frac{1}{3} \\ &= 2\frac{1}{3}. \end{aligned}$$

The solution for the system of simultaneous equations $y = 3 - 2x$ and $y = 2 + x$ is the point

$$\left(\frac{1}{3}; 2\frac{1}{3}\right).$$

Activity

Suppose we have the system of simultaneous equations

$$2x + 5y = 13 \text{ and } 3x + 4y = 9.$$

Rewrite the equations to make y the subject of the expression and solve.

Answer

If

$$2x + 5y = 13,$$

then

$$\begin{aligned} 5y &= 13 - 2x \\ y &= \frac{13}{5} - \frac{2}{5}x. \end{aligned}$$

If

$$3x + 4y = 9,$$

then

$$\begin{aligned} 4y &= 9 - 3x \\ y &= \frac{9}{4} - \frac{3}{4}x. \end{aligned}$$

Now we have

$$\begin{aligned} \frac{13}{5} - \frac{2}{5}x &= \frac{9}{4} - \frac{3}{4}x \\ \frac{13}{5} - \frac{9}{4} &= \frac{2}{5}x - \frac{3}{4}x \\ \frac{52-45}{20} &= \frac{8-15}{20}x \\ \frac{7}{20} &= -\frac{7x}{20} \\ 7 &= -7x \\ x &= -1. \end{aligned}$$

Substitute $x = -1$ into $y = \frac{13}{5} - \frac{2}{5}x$:

$$\begin{aligned} y &= \frac{13}{5} - \frac{2}{5} \times -1 \\ &= \frac{13}{5} + \frac{2}{5} \\ &= \frac{15}{5} \\ &= 3. \end{aligned}$$

The solution for the system of simultaneous equations is

$$(-1;\ 3).$$

An alternative method is the following:

Consider the system of simultaneous equations

$$\begin{aligned} 2x + 5y &= 13 \quad (1) \\ 3x + 4y &= 9. \quad (2) \end{aligned}$$

From the first equation or (1) we have

$$\begin{aligned} 2x &= 13 - 5y \\ x &= \frac{13}{2} - \frac{5}{2}y. \quad (3) \end{aligned}$$

Substitute x or (3) into the second equation or (2):

$$\begin{aligned} 3\left(\frac{13}{2} - \frac{5}{2}y\right) + 4y &= 9 \\ \frac{39}{2} - \frac{15}{2}y + \frac{8}{2}y &= 9 \\ \frac{39}{2} - \frac{7}{2}y &= 9 \\ -\frac{7}{2}y &= \frac{18}{2} - \frac{39}{2} \\ -\frac{7}{2}y &= -\frac{21}{2} \\ -7y &= -21 \\ y &= 3. \end{aligned}$$

Substitute $y = 3$ into (3) or $x = \frac{13}{2} - \frac{5}{2}y$:

$$\begin{aligned} x &= \frac{13}{2} - \frac{5}{2} \times 3 \\ &= \frac{13}{2} - \frac{15}{2} \\ &= \frac{-2}{2} \\ &= -1. \end{aligned}$$

We have the same solution obtained previously!

Exercise 5.1

Solve the following systems of equations:

1. $7x + 5y = -4$ and $3x + 4y = 2$
2. $2x + 2y = 3$ and $5x + \frac{1}{2}y = -6$
3. $x + 4y = 49$ and $-2x + y = 1$

Study unit 5.3 Linear inequalities in one variable

Learning objectives: *On completion of this study unit you should be able to solve a linear inequality in one variable.*

You are already familiar with the concepts "**greater than**", which is denoted by the symbol $>$, and "**less than**" which is denoted by the symbol $<$.

We say that ***b* is greater than *a*** if and only if the difference $b - a$ is positive. Thus,

$$b > a$$

means

$$b - a > 0.$$

We say that ***b* is less than *a*** if and only if the difference $b - a$ is negative. Thus,

$$b < a$$

means

$$b - a < 0.$$

If $b = a$ or $a = b$ then a and b are the same point.

Sometimes the symbol $=$ is combined with $>$ or $<$ as follows:

$$b \geq a \text{ means } b - a \geq 0,$$

$$b \leq a \text{ means } b - a \leq 0.$$

If we replace the $=$ sign in any equation, or system of equations, with one of $>$, $<$, $\geq$ or $\leq$, then we obtain an **inequality** or **system of inequalities**. The inequality, or system of inequalities, is linear or non-linear as the functions involved are linear or non-linear.

To solve an inequality means to find all values of the variable for which the inequality is true.

For example, if the inequality $5x - 15 < 0$ had to be solved, it means that you have to determine the values of x for which this statement is true. How is this done?

First we note that

$$b > a$$

implies

$$a < b$$

and vice versa. That is, if ***b* is greater than *a*, then *a* is less than *b***. (Eg $4 > 1$ implies $1 < 4$). Although you might say that this is obvious, it is nevertheless very useful at times when we want to interchange the left- and right-hand sides.

Secondly, if

$$b > a,$$

then

$$b + c > a + c$$

for any number c.

That is, we may add the same number to both sides of an inequality. For example, since

$$4 > 1$$

then

$$4 + 3 > 1 + 3,$$

that is,

$$7 > 4$$

and

$$4 - 5 > 1 - 5,$$

that is,

$$-1 > -4.$$

Thirdly, if

$$b > a$$

and

$$c > 0$$

then

$$b \times c > a \times c$$

and

$$b \div c > a \div c.$$

That is, we may multiply or divide both sides of an inequality by the same positive number. For example, since

$$4 > 1$$

thus

$$4 \times 8 > 1 \times 8,$$

that is,

$$32 > 8$$

and

$$4 \div 20 > 1 \div 20,$$

that is,

$$0{,}2 > 0{,}05.$$

Fourthly, if

$$b > a$$

but

$$c < 0$$

then

$$b \times c < a \times c$$

and

$$b \div c < a \div c.$$

That is, we may multiply or divide both sides of an inequality, by the same negative number, but then we must reverse the sense of the inequality that is, change $>$ to $<$, or change $<$ to $>$. For example, since

$$4 > 1$$

then

$$4 \times -5 < 1 \times -5,$$

that is,

$$-20 < -5$$

and

$$4 \div -8 < 1 \div -8,$$

that is,

$$-0{,}5 < -0{,}125.$$

These are the four rules that we need to solve linear inequalities. Summarised, they are, if

$$b > a$$

then:

1. $a < b$
2. $b + c > a + c$ for any c
3. $b \times c > a \times c$ and $b \div c > a \div c$ for any $c > 0$
4. $b \times c < a \times c$ and $b \div c < a \div c$ for any $c < 0$

Examples

1. Solve

   $$5x - 15 < 0.$$

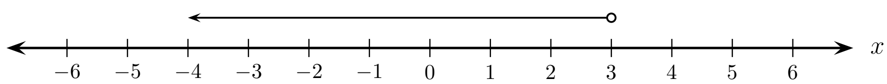
   Add +15:

   $$5x < 15.$$

   Divide by 5:

   $$x < 3.$$

   This is indicated on the line by highlighting all points for which the inequality is true. Since the point $x = 3$ is not included, it is shown by means of an open circle.

2. Solve

   $$5x + 6 \geq 6x - 5.$$

   Subtract 6:

   $$5x \geq 6x - 11.$$

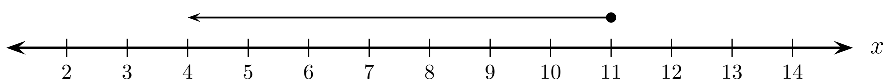
   Subtract $6x$:

   $$\begin{aligned} 5x - 6x &\geq -11 \\ -x &\geq -11. \end{aligned}$$

   Multiply by −1:

   $$x \leq 11.$$

   The solution consists of all real values less or equal to 11. This is indicated on the line below:

Exercise 5.2

Solve the following:

1. $11 \geq 6 - 4x$
2. $4x + 4 < 1{,}5x - 6$

Study unit 5.4 Systems of linear inequalities in two variables

Learning objectives: *On completion of this study unit you should be able to solve a system of linear inequalities in two variables.*

5.4.1 Linear inequalities in two variables

Although the rules in study unit 3.3 may be applied to linear inequalities in two, or more variables, they are not of much use when it comes to solving a **system of linear inequalities** in two or more variables. The trouble is that the solution is not generally a single point but usually an infinite sequence of points. In fact, in the two variable cases it is usually even more than that – it is a whole area, or region as it is known, in the x-y plane. This means that the most successful approach to solving linear inequalities in two variables is by means of graphs.

Suppose we have to draw a graph with a set of points that obey the inequality

$$-x + 2y - 2 \geq 0.$$

Now, if we ignore the $>$ sign for a moment and just consider the $=$ sign, we can easily draw a graph of

$$-x + 2y - 2 = 0.$$

This is the straight line $y = \frac{1}{2}x + 1$ depicted in figure 5.4.1.

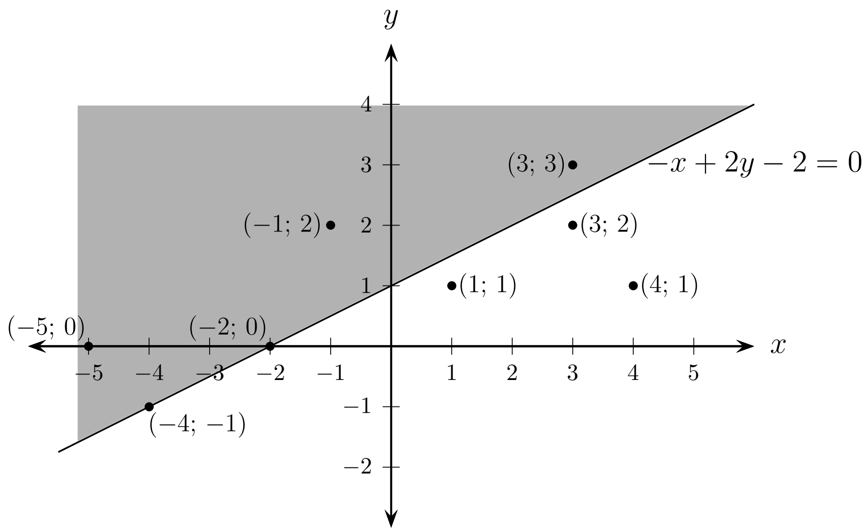


Figure 5.4.1

Now any point on the line satisfies the original inequality (or rather the $=$ part of it). Thus, all points on the line are in the solution space. What about points not on the line? Well, the simple approach is to simply test a few and see whether they satisfy the inequality or not.

Consider (1; 1). Substitute it into the left-hand side of the inequality, $-x + 2y - 2 \geq 0$. It does not satisfy the inequality since

$$-1 + 2 \times 1 - 2 = -1$$

and

$$-1 \ngeq 0$$

(read this as "is not greater than or equal to 0"). Also note that it lies below the straight line on the graph. So, too, do the points $(4; 1)$ and $(3; 2)$. Nor do they satisfy the inequality. On the other hand, the points $(3; 3)$, $(-1; 2)$ and $(-5; 0)$ all satisfy the inequality and all lie above the straight line. For example, when $(3; 3)$ is substituted into the left-hand side of the inequality, $-x + 2y - 2 \geq 0$, we find

$$-3 + 2 \times 3 - 2 = 1$$

and

$$1 \geq 0.$$

If we carry on in this fashion we will find that all points above the line satisfy the inequality, whereas all points below the line do not. Thus, the line divides the x-y plane into two regions – those that satisfy the inequality and those that do not. In this case the points on the line, for example $(-2; 0)$ and $(-4; -1)$, also satisfy the inequality (since the $=$ sign is included in the statement). We can use this result as the basis of a prescription for graphing inequalities.

Rules for graphing linear inequalities:

1. Graph the line that results when the inequality is changed to an equality.
2. Select any point not on the line.
3. If the coordinates of the point satisfy the inequality, then all points on the same side of the line satisfy the inequality.
4. If the coordinates of the point do not satisfy the inequality, then all points on the opposite side of the line satisfy the inequality.
5. If the $=$ sign is part of the inequality, then the points on the line also satisfy the inequality, otherwise they do not.

This means that all you have to do in order to solve a single inequality in two variables is to draw the corresponding straight line and to examine a single point in the plane.

Exercise 5.3

Draw the following linear inequalities on a graph:

1. $3x + y - 3 > 0$
2. $2x + 4y + 1 \leq x + y - 2$

5.4.2 Systems of linear inequalities in two variables

Just as systems of linear equations can be formulated, so can systems of linear inequalities. When solving a system we must determine all points that simultaneously satisfy all linear inequalities in the system.

A system of inequalities means that there is more than one line involved. The lines must be drawn on *one* set of axes and therefore there is *only one* solution space.

Once again, the solution is generally a region in the x-y plane. This is demonstrated by the example below.

Consider the system of inequalities:

$$-x + y - 1 \leq 0 \quad \text{and} \quad 2x + y - 4 < 0.$$

If we examine each inequality separately we can graph its solution along the lines discussed in the previous study unit. The solution of the first inequality is the region including all points on the line $-x + y - 1 = 0$ and all those below and to the right of this line. The solution of the second inequality is all points below and to the left of the line $2x + y - 4 = 0$, but not the points on the line.

The two solutions are depicted graphically in figure 5.4.2, the first with vertical lines and the second with horizontal lines.

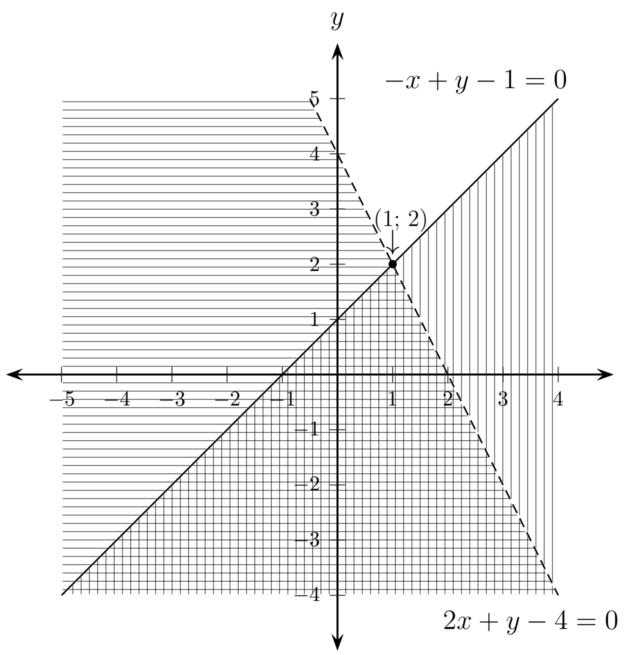


Figure 5.4.2

Now the region in the figure where the horizontal and vertical lines cross, the so-called cross-hatched or grey space, is the region in which both inequalities are satisfied. It is, in other words, **the solution of the system of inequalities**. Note that the first line is included in the solution while the second is not, as is indicated by dashing the second line. An important point is the corner of the region where the two lines intersect at $(1; 2)$. This is known as an **extreme point** of the region of solution.

To determine the solution of a system of inequalities, solve each inequality separately and determine the region that is common to all solutions. This region is known as the solution space of the system of inequalities.

Activity

Solve the system of inequalities graphically:

$$x + y + 2 \geq 0 \text{ and } -x + 2y + 2 < 0$$

Answer

The inequalities are

$$x + y + 2 \geq 0 \text{ and } -x + 2y + 2 < 0.$$

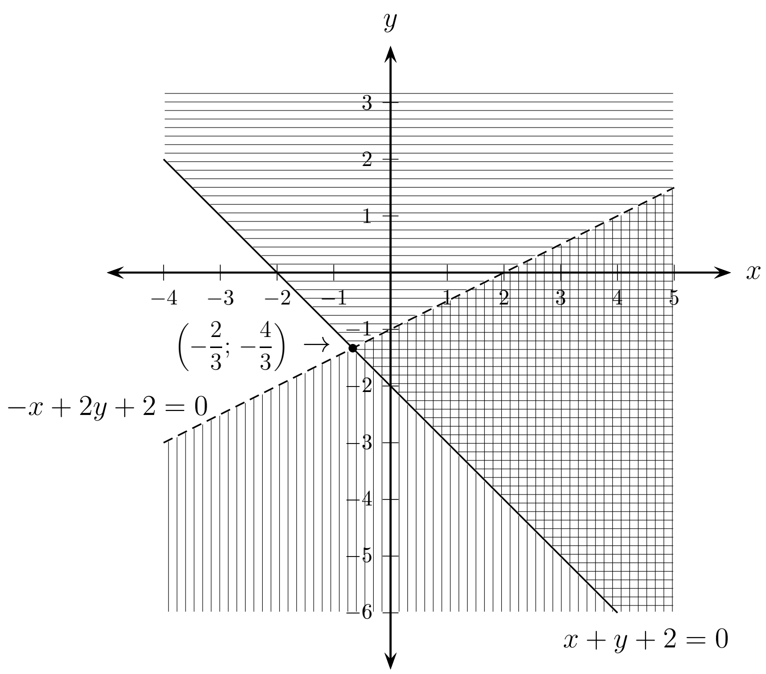
The solution of the first inequality is the region including all the points on, above and to the right of the line

$$x + y + 2 = 0.$$

This is the region indicated by the horizontal lines in the sketch below. The solution of the second inequality is the region including all points below and to the right of the line

$$-x + 2y + 2 = 0,$$

but not the points on the line. This region is indicated by the vertical lines in the sketch below. The solution of the system of two inequalities is the cross-hatched region.



So far we have only considered systems of two inequalities in two unknowns. However, it is quite feasible, and very often necessary in practice, to consider systems with a greater number of inequalities than the number of unknowns. This is illustrated below, where we consider a system of five inequalities in two unknowns. The procedure for solving the system is exactly as before, namely, solve each inequality separately and then determine which region is common to all solutions.

Activity

Solve the system of inequalities graphically:

$$\begin{aligned} -x+y-3 &\leq 0 \\ x+y-5 &\leq 0 \\ x-3 &\leq 0 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

Answer

We look at the last two of these inequalities first. The inequality $x \geq 0$ simply means the region to the right of and including the y-axis, whereas $y \geq 0$ means the region above and including the x-axis. Taken together, these two inequalities imply the first quadrant of the x-y plane so we can restrict our considerations to this region.

Now the first inequality is satisfied by all points on, below and to the right of the line

$$-x+y-3=0,$$

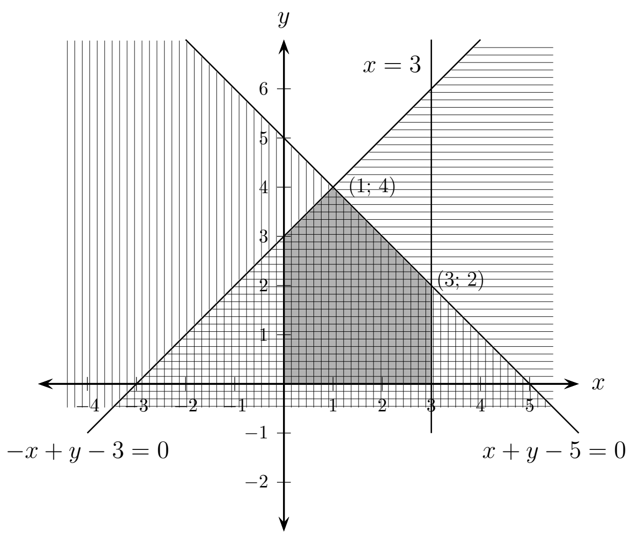
that is, the region shaded with horizontal lines. The second is satisfied by all points on, below and to the left of the line

$$x+y-5=0,$$

that is, the region shaded with vertical lines. Finally, the inequality

$$x-3 \leq 0 \quad \text{or} \quad x \leq 3$$

is satisfied by all points on and to the left of the vertical line $x = 3$. The solution is the cross-hatched area shown in graph below.



Exercise 5.4

1. Solve the following system of inequalities graphically:

$$\begin{aligned} 2x + y - 5 &\leq 0 \\ x - 2 &\leq 0 \\ y - 4 &\leq 0 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

2. Solve

$$3x - 7 \leq 5x + 2$$

and indicate your solution on the number line.

3. Solve

$$5x + y + 1 < -x - y - 1$$

graphically.

COMPONENT 6

An application of differentiation

After completion of this component you should be able to apply some rules of differentiation to calculate marginal costs and marginal profits and to interpret these.

CONTENTS

Study unit 6.1 Marginal profit

Learning objectives: *On completion of this study unit you should be able to calculate the marginal profit at a specific production level.*

Few things in the business world are more important than the study of change; change in the sales of a company; change in the value of the rand; change in the value of shares; change in the interest rate and so forth.

Equally important is the **rate** at which these changes take place. If the sales of a company increased by R2 000 000, it is important to know whether this change occurred over one year, two years or ten years. If the consumer price index went up by 0,5%, it does not mean much unless we know whether this change occurred over a month or over a year. This rate of change need not refer to changes over time only, but can also be something like the change in costs for different production quantities in a production process.

Change consists of two components: size and direction.

Do you remember the linear functions discussed in component 2?

The equation for a linear function, or straight line, is

$$y = ax + b$$

where a is the slope of the line and b is the intercept of the line on the y-axis.

If $(x_1; y_1)$ and $(x_2; y_2)$ are two points on a straight line then the slope is defined as

$$a = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

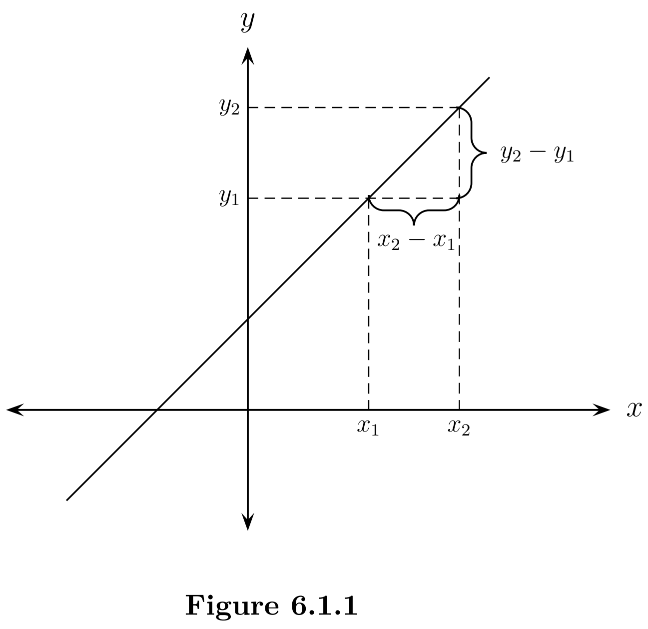


Figure 6.1.1

The slope is the change in y which corresponds to a change of one unit in the value of x. The value of the slope gives the size of the change, while the sign of the slope gives the direction of the change.

If the slope has a positive sign it indicates an increase; if the slope has a negative sign it indicates a decrease.

When the function is not linear but, for example, a quadratic function, the size and direction are not constant, but change continuously.

The following example illustrates this:

A maize farmer's cost function in rand, is given by

$$C(x) = \frac{x^2}{10} + 100$$

where x represents the number of tons of mealies produced. The cost is in rand.

The revenue function is given by

$$R(x) = 250x,$$

that is, each ton of mealies produced is sold at R250,00.

His profit is the difference between income earned by selling (revenue) and the cost incurred for the production of the maize.

The profit is now given by

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 250x - \left(\frac{x^2}{10} + 100\right) \\ &= 250x - \frac{x^2}{10} - 100. \end{aligned}$$

A graph of the profit function is given in figure 6.1.2.

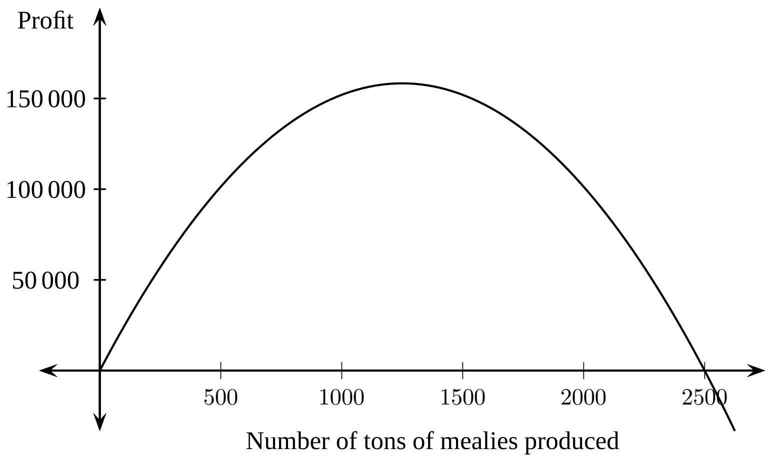


Figure 6.1.2

We can clearly see that the profit increases to a point where the profit is at a maximum, after which the profit decreases. The slope of the function is not constant and there is no constant rate of change.

The farmer can see where the profit will increase and where it will decrease, but he cannot see how the change in production level will influence the profit. What he wants to determine is the **marginal profit** at a specific production level.

The marginal profit is the change in profit resulting from a change in the number of units produced.

This means that if the farmer produces 1 000 tons and he is interested in knowing how the profit will change if one extra ton is produced, then he wants to determine the marginal profit at a production level of 1 000 tons.

In non-mathematical books this is written as

$$\begin{aligned}\text{marginal profit} &= \frac{\text{change in profit}}{\text{change in number of units}} \\ &= \text{slope of the profit function.}\end{aligned}$$

In order to know the change in profit at different production levels, the slope of the function at the different production levels must be determined. The method used in mathematics to determine this is called differentiation.

In general, the slope of the function $y = f(x)$ is referred to as the derivative of the function and is written as

$$\frac{dy}{dx}.$$

We pronounce this as "dee-y-dee-x". Another notation is

$$f'(x).$$

The derivative gives both the size and the direction of the change.

There are rules of differentiation that can be used to obtain the derivative. The derivation of the rules is beyond the scope of this module and I will just give you some of the rules of differentiation.

There are a number of differentiation rules, but we will consider only the following four:

1. The derivative of any constant term, that is, a term which consists of a number only, is zero:

   $$\frac{da}{dx} = 0,$$

   where a is a constant.

   Example: If $f(x) = 4$ then $f'(x) = 0$.

2. If $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

   Read as: If $f(x)$ equals x to the power of n, then $f'(x)$ (pronounce as: f accent x) is equal to n times x to the power of n minus one.

   Example: If $f(x) = x^7$ then $f'(x) = 7x^6$.

3. $\frac{d}{dx}[af(x)] = af'(x)$.

   Read as: The derivative of a constant times a function is equal to the constant times the derivative of the function.

   If $f(x) = ax^n$, then $f'(x) = anx^{n-1}$.

   Example: If $f(x) = 7x^5$ then $f'(x) = 7 \times 5x^4 = 35x^4$.

4. If $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$.

   Read as: When a function consists of the sum of two other functions, then the derivative of this function is the sum of the derivatives of the other two functions.

   If $f(x) = a + g(x)$, then $f'(x) = g'(x)$.

   Read as: When a function consists of the sum of a constant and another function, then the derivative of this function is zero plus the derivative of the other function.

   Example: If $f(x) = x^2 - 2x - 3$ then $f'(x) = 2x - 2$.

Activity

Use a table to illustrate how the rules are applied for

$$f(x) = x^n$$

for $n = 0; 1; 2; 3$ and 4.

Answer

Application of the rules gives

$f(x)$	$f'(x)$
$x^0 = 1$	0
x	1
x^2	$2x$
x^3	$3x^2$
x^4	$4x^3$.

If the farmer produces 1 000 tons of mealies and he is interested in knowing how the profit will change if one extra ton is produced, then he must determine the marginal profit at a production level of 1 000 tons.

He uses the following procedure:

$$P(x) = 250x - \frac{x^2}{10} - 100.$$

Determine the derivative of each term:

$$\begin{aligned} \frac{d}{dx}250x &= 250x^{1-1} \\ &= 250x^0 \\ &= 250 \\ \frac{d}{dx}\left(-\frac{x^2}{10}\right) &= -\frac{2x^{2-1}}{10} \\ &= -\frac{x}{5} \\ \frac{d}{dx}100 &= 0 \end{aligned}$$

The derivative of the profit function

$$P(x) = 250x - \frac{x^2}{10} - 100$$

is then

$$P'(x) = 250 - \frac{x}{5}.$$

Substitute $x = 1\,000$ into $P'(x) = 250 - \frac{x}{5}$ to calculate the marginal profit at a production level of 1 000 tons:

$$\begin{aligned} P'(1\,000) &= 250 - \frac{1\,000}{5} \\ &= 50. \end{aligned}$$

For an extra ton produced at a production level of 1 000 tons, the profit will increase by R50,00 per ton.

Activity

Calculate the farmer's marginal profit at production levels of 500, 1 500 and 1 250 tons respectively and interpret it.

Answer

If

$$P(x) = 250x - \frac{x^2}{10} - 100,$$

then

$$P'(x) = 250 - \frac{x}{5}$$

and

$$\begin{aligned} P'(500) &= 250 - \frac{500}{5} \\ &= 150. \end{aligned}$$

At a production level of 500 tons the profit will increase by R150,00 per ton for an increase in production.

For a production level of 1 500 tons:

$$\begin{aligned} P'(1\,500) &= 250 - \frac{1\,500}{5} \\ &= -50. \end{aligned}$$

At a production level of 1 500 tons the profit will decrease by R50,00 per ton for an increase in production.

For a production level of 1 250 tons:

$$\begin{aligned} P'(1\,250) &= 250 - \frac{1\,250}{5} \\ &= 0. \end{aligned}$$

At first the marginal profit is positive and the profit increases for every unit more produced, but then there is a turning point and at 1 500 tons the marginal profit is negative. The negative marginal profit indicates that any extra production will yield a decrease in the profit.

The turning point, or the value where the profit will be a maximum, lies somewhere between 1 000 and 1 500 tons. At this point, where the profit is at a maximum, the marginal profit is equal to zero. The slope of the profit function at the turning point is zero – the marginal profit will thus be zero where the profit is maximised.

Thus, the farmer must produce 1 250 tons to yield the maximum profit. Let us check this.

We know that

$$P'(x) = 250 - \frac{x}{5}.$$

Let $P'(x) = 0$. Then

$$\begin{aligned} 250 - \frac{x}{5} &= 0 \\ \frac{x}{5} &= 250 \\ x &= 250 \times 5 \\ &= 1\,250 \end{aligned}$$

which is the same answer as previously determined.

Although the farmer might have the capacity to increase production, the cost function is such that maximum production does not necessarily indicate maximum profit.

Exercise 6.1

Calculate the derivatives of the following functions:

1. $f(x) = 6x$
2. $f(x) = 3 + 5x$
3. $f(x) = 3 + 2x^2$
4. The profit (in rand) yielded by the sales of x knives is given by the function

$$P(x) = 5x - \frac{x^2}{200} - 450.$$

Calculate the marginal profit at $x = 450$ and $x = 750$ and interpret the results.

5. A manufacturer of scissors estimates his revenue function (in rand) to be

$$R(x) = 10x - \frac{x^2}{1\,000},$$

where x is the number of scissors produced. The cost function (in rand) is

$$C(x) = 7\,000 + 2x.$$

Calculate and interpret the marginal profit when 2 000, 4 000 and 5 000 scissors are produced.

Remember:

To find the maximum profit, let $P'(x) = 0$ and solve for x;
at the x-value where $P'(x) = 0$, profit is maximised.

Study unit 6.2 Marginal costs

Learning objectives: *On completion of this study unit you should be able to determine the marginal cost at a specific production level and to interpret it.*

The concept of marginal cost is a very important one in the theory of economics. The marginal cost is the change in cost as a result of a change in the number of units produced.

In non-mathematical books this is written as

$$\begin{aligned} \text{MC} &= \text{marginal cost} \\ &= \frac{\text{change in cost}}{\text{change in number of units produced}} \\ &= \text{slope of the cost function.} \end{aligned}$$

Using the knowledge obtained in study unit 7.1 we write

$$\begin{aligned} \text{MC} &= \text{marginal cost} \\ &= \frac{dC}{dx} \\ &= C'(x). \end{aligned}$$

Consider the case of the maize farmer in the previous study unit. The cost function in rand is given by

$$C(x) = \frac{x^2}{10} + 100$$

where x represents the number of tons of mealies produced.

A graph of the cost function is given in figure 6.2.1.

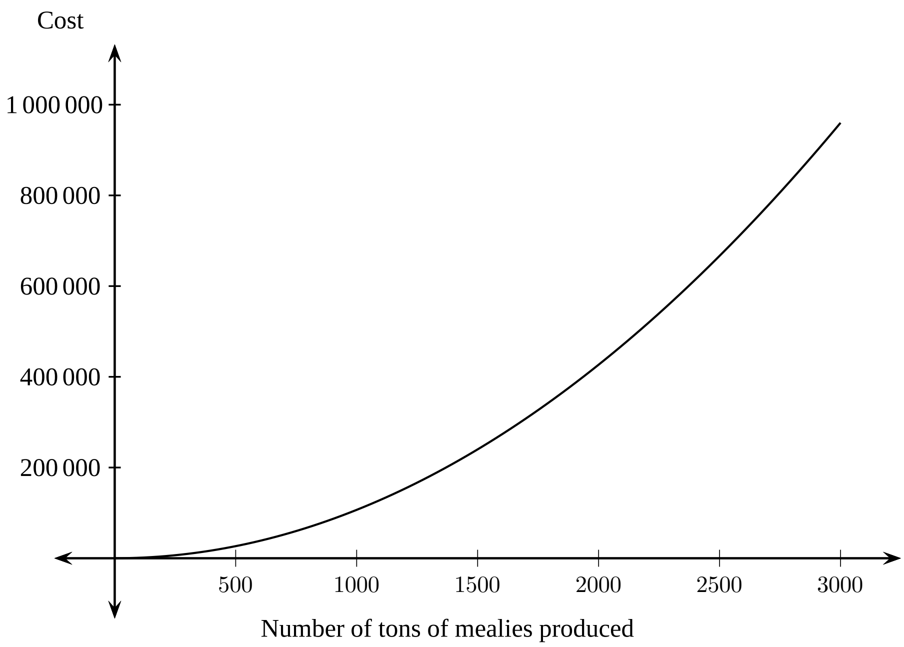


Figure 6.2.1

We can clearly see that the cost function is continuously increasing until it reaches a point where it will be extremely expensive for him to produce another ton of mealies. The slope of the function is not constant and there is no constant rate of change. Thus the slope changes continuously.

The farmer can see how the cost is increasing but he cannot see how a change in production level will influence the cost. In other words, if he produces at a certain production level, what will it cost to produce one extra ton?

For instance, if the farmer produces 1 000 tons of mealies and he is interested in knowing how the cost will change if one extra ton is produced, then he wants to determine the marginal cost at a production level of 1 000 tons.

He does this by carrying out the following procedure:

$$C(x) = \frac{x^2}{10} + 100.$$

Determine the derivative of each term:

$$\begin{aligned} \frac{d}{dx}\left(\frac{x^2}{10}\right) &= \frac{2x^{2-1}}{10} \\ &= \frac{x}{5} \\ \frac{d}{dx}100 &= 0 \end{aligned}$$

The derivative of the cost function $C(x) = \frac{x^2}{10} + 100$ is then

$$C'(x) = \frac{x}{5}.$$

Substitute $x = 1\,000$ into $C'(x) = \frac{x}{5}$ to calculate the marginal cost at a production level of 1 000 tons:

$$\begin{aligned} C'(1\,000) &= \frac{1\,000}{5} \\ &= 200. \end{aligned}$$

For an extra ton produced at a production level of 1 000 tons, the cost will increase by R200,00 per ton.

Activity

For the maize farmer, calculate the cost function and the marginal cost for the production of 2 000, 3 000 and 4 000 tons. Interpret the various marginal costs.

Answer

In the following table the cost function is given by $C(x)$ and the marginal cost function by $C'(x)$.

Tons (x)	$C(x) = \frac{x^2}{10} + 100$	$C'(x) = \frac{x}{5}$
2 000	400 100	400
3 000	900 100	600
4 000	1 600 100	800

If we now consider the marginal cost, we see that if the farmer produces 2 000 tons, then it will cost him an extra R400 per ton to produce the next ton.

If, however, the farmer produces 3 000 tons, then it will cost him an additional R600 per ton to produce the next ton.

If the farmer produces 4 000 tons, it will cost him an extra R800 per ton to produce the next ton.

Activity

Sweet Tooth, which specialises in homemade chocolates, packs the chocolates in 100 g gift packets. The fixed production cost is R200,00 per day while the production cost per 100 g packet amounts to a further R9,00 per packet. Sweet Tooth cannot produce more than 200 packets per day. What is the marginal cost if Sweet Tooth produces 150 packets per day? How do you interpret this value?

Answer

Let x represent the number of packets produced per day.

The production cost per day is

$$\text{fixed cost per day} + \text{variable cost.}$$

The value 200 represents the **fixed cost**, in other words the cost incurred before a single packet of chocolates is produced.

The **variable cost** may be described as the cost per unit multiplied by the number of units produced. Therefore, the variable cost is

$$\text{cost per packet} \times \text{number of packets}$$

or in this case

$$9 \times x.$$

The cost function is therefore

$$C(x) = 200 + 9x.$$

To determine the marginal cost we have to calculate $C'(x)$:

$$\begin{aligned} C'(x) &= 0 + 9 \\ &= 9. \end{aligned}$$

The derivative of the function is 9 and the marginal cost is also 9 (which is a constant).

For a straight line the slope remains constant. Therefore the marginal cost remains constant.

As we have seen when we discussed marginal profit, it is another matter when the function is not a straight line. Then the slope changes continuously.

Let us consider the cost function of the maize farmer in Study unit 7.1 again.

The cost function in rand, is given by

$$C(x) = \frac{x^2}{10} + 100$$

where x represents the number of tons of mealies produced.

The marginal cost function is

$$\frac{dC}{dx} = C'(x) = \frac{2x}{10} + 0 = \frac{x}{5}.$$

As seen in the activity on page 169, we see that if the farmer produces 2 000 tons, then it will cost him an extra R400 per ton to produce the next ton. If, however, the farmer produces 3 000 tons, then it will cost him an additional R600 per ton to produce the next ton. If the farmer produces 4 000 tons, it will cost him an extra R800 per ton to produce the next ton.

His cost function is such that it is constantly increasing until it reaches a point where it will be extremely expensive for him to produce another ton of mealies.

Exercise 6.2

The total cost $C(x)$, in thousands of rand, to manufacture x small sailing boats is given by the function

$$C(x) = 575 + 25x - \frac{x^2}{4}.$$

1. Calculate the marginal cost if x boats are manufactured.
2. Calculate the marginal cost if 40 boats are manufactured and interpret the result.
3. Calculate $C'(30)$ and interpret it.

COMPONENT 7

Mathematics of finance

On completion of this component you should be able to

- calculate simple interest and compound interest
- determine the present value and future value of an annuity
- set up an amortisation schedule for a loan
- reschedule payments on a loan for changes in interest rate or term

CONTENTS

Study unit 7.1 Simple interest and simple discount

Learning objectives: *On completion of this study unit you should*

- *calculate simple interest;*
- *manipulate the formula in order to obtain expressions for the respective variables and*
- *calculate simple discount.*

A very important class of formulae is the one related to so-called interest calculations.

(Please note: The derivation of the formulae is beyond the scope of this module and the formulae will only be given.)

Interest is the price paid for the use of borrowed money. As money is important, let us start immediately!

7.1.1 Simple interest

Interest is paid by the party who uses or borrows the money to the party who lends the money. Interest is calculated as a fraction of the amount borrowed or saved (principal amount) in a time period. The fraction, also known as the interest rate is usually expressed as a percentage per year, but must be reduced to a decimal fraction for calculational purposes. For example, if we have borrowed an amount from the bank for a period of one year at an interest rate of 12% per year, we can express the interest as

$$12\% \text{ of the amount borrowed} \times \text{number of periods borrowed,}$$

that is,

$$\frac{12}{100} \text{ of the amount borrowed} \times \text{one year,}$$

that is,

$$0{,}12 \times \text{the amount borrowed} \times \text{one year.}$$

When and how interest is calculated gives way to different types of interest. For example, simple interest is interest which is calculated at the end of the entire term for which the money was used, on the original amount (principal amount) used (borrowed or saved).

Now let's look at an example.

How much simple interest will be paid on a loan of R10 000 borrowed for a year at an interest rate of 15% per year?

Now 15% of the R10 000 must be paid as interest per year for the use of the money. Thus, the interest per year is calculated as

$$0{,}15 \times 10\,000 = 1\,500.$$

The interest per year is R1 500.

Suppose we are using the money for two years. Thus, the interest for the full loan period or term is

$$1\,500 \times 2 = 3\,000.$$

The interest for the full load period is thus R3 000. In this case we have multiplied the amount used by the interest rate per year, multiplied by the number of years used.

Simple interest is interest which is computed, on the principal, for the entire term for which it is borrowed, and which is therefore due at the end of the term and is given by

$$I = PRT$$

where

I	is the **simple interest** (in rand) paid for the use of the money at the end of the term
P	is the **principal**, or total amount borrowed (in rand) which is subject to interest (this is also known as the **present value** (***PV***) of the loan)
R	is the **rate of interest**, that is, the fraction of the principal which must be paid at the end of **each period** (say, year) for the use of the principal
T	is the **term** or time, that is, the number of periods for which the principal is borrowed

Note

1. The units used for the rate of interest and the term must be consistent, that is, if the interest rate is per annum, then the term must be in years, or a fraction thereof. If the interest rate is expressed for a shorter period (say, per month) then the term must be expressed accordingly (ie in months). It should be pointed out here that, in this respect, a distinction is sometimes made between the so-called **ordinary interest year**, which is based on a 360-day year, and the **exact interest year**, which is based on a 365-day year. In the former case, each month has 30 days, each quarter 90 days and so on. In the latter, the exact number of days or months of the loan is used. Unless otherwise stated, we shall always work with exact periods.

2. In other textbooks, you may find that, different symbols are used for the different variables. By now this should not be a source of confusion to you.

The **amount** or **sum accumulated** ***S***, also known as the **maturity value**, **future value** or **accrued principal** at the end of the term T, is given by

$$\begin{aligned} S &= \text{principal value} + \text{interest} \\ S &= P + I \\ S &= P + PRT \\ S &= P(1 + RT). \end{aligned}$$

The date at the end of the term on which the debt is to be paid is known as the **due date** or **maturity date**.

Sometimes, we not only consider the basic formula

$$I = PRT,$$

but also turn it inside out and upside down, as it were, in order to obtain formulae for each variable in terms of the others.

Of particular importance is the concept of **present value**, P or PV, which is obtained from the basic formula for the sum or future value S, namely

$$S = P \times (1 + RT).$$

Dividing by the factor $(1 + RT)$ gives

$$P = S \div (1 + RT).$$

How do we interpret this result? As follows: P is the amount which must be borrowed now to accrue to the sum S after a term T at interest rate R per time unit. As such, it is known as the present value of the sum S. Stated formally, the **present value** of a debt S on a date prior to the due date is the value P or PV of the debt on the date in question, and is given by the formula

$$P = S \div (1 + RT)$$

where R is the rate of interest and T the "time to run to maturity".

Activity

Obtain an expression for T, the term, from the formula

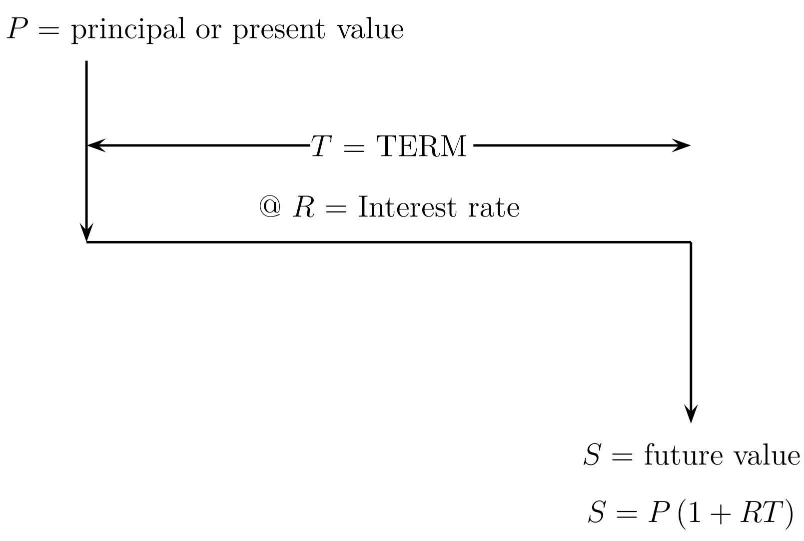
$$S = P \times (1 + RT).$$

Answer

This is done as follows:

$$\begin{aligned} S &= P \times (1 + RT) \\ S &= P + PRT \\ S - P &= PRT \\ \frac{S - P}{PR} &= \frac{PRT}{PR} \\ \frac{S - P}{PR} &= T \\ T &= \frac{S - P}{PR} \end{aligned}$$

A very useful way of representing interest rate calculations is with the aid of a so-called **time line**. For a simple interest rate calculation, the time line is as follows:



(The symbol @ is used for "at".)

At the beginning of the term, the principal P (or present value) is put down (or borrowed) – that is, is entered onto the line. At the end of the term, the amount or sum accumulated S (or future value) is received (or paid back). Note that the sum accumulated includes the interest received.

The sum accumulated is calculated as

$$\begin{aligned} \text{sum accumulated} &= \text{principal} + \text{interest received} \\ S &= P + PRT \\ S &= P \times (1 + RT). \end{aligned}$$

Exercise 7.1

1. Calculate the simple interest and sum accumulated when R5 000 is borrowed for 90 days at 12% simple interest per annum.
2. Determine the principal required to yield R300 in 18 months at $12\frac{1}{2}$% simple interest per annum. (Remember: *yield = return on money = interest earned*.)
3. How much must be paid back on a loan of R3 000 for six months if simple interest of 12% per annum is charged?
4. You invest R1 000 at a simple interest rate of 10% per annum for four years. What is the total interest that you receive? If the interest is paid monthly how much do you receive each month?
5. You borrow R800 and three months later pay back R823 in full settlement of your debt. What is the annual simple interest charged?

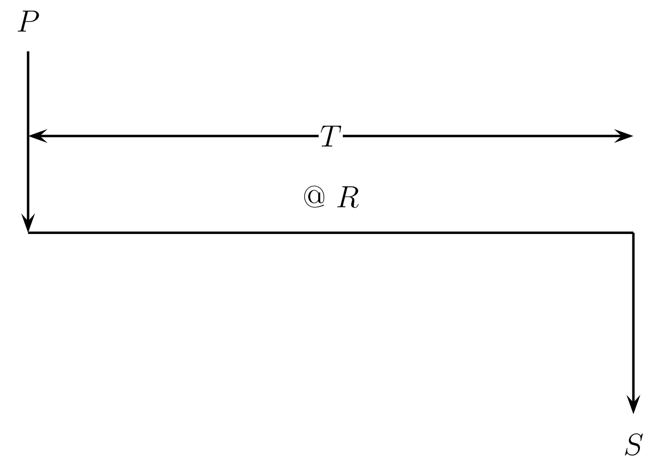
7.1.2 Simple discount

Interest calculated on the principal value but payable at the beginning of the term is called **discount**.

In the previous paragraph emphasis was placed on the interest which has to be paid at the end of the term for which the loan (or investment) is made. On the due date, the principal borrowed **plus** the interest earned is paid back (or received).

In practice, there is no reason why the interest cannot be paid at the beginning rather than at the end of the term. Indeed, this in effect implies that the lender deducts the interest from the principal in advance. At the end of the term, only the principal is then due. Loans handled in this way are said to be **discounted**, and the interest paid in advance is called the **discount**. Also the amount then advanced by the lender is termed the **discounted value**. Clearly, the discounted value is simply the present value of the sum to be paid back.

Expressed in terms of the time line of the previous paragraph this means that we are given S and are asked to calculate P.



(The symbol @ is used for "at".)

The discount, on the sum S, is then simply the difference between the future and present values. Thus the discount (D) is given by

$$D = S - P.$$

We could continue and substitute $P = S \div (1 + RT)$ in the above formula to obtain an expression for D in terms of S, R and T. In practice, however, this is not usually the way in which financial institutions quote discounted instruments. Rather, they introduce a simple discount rate, d, which, by analogy with the interest rate, is the fraction of the sum S per time period which must be paid. The rate, d, is expressed as a percentage.

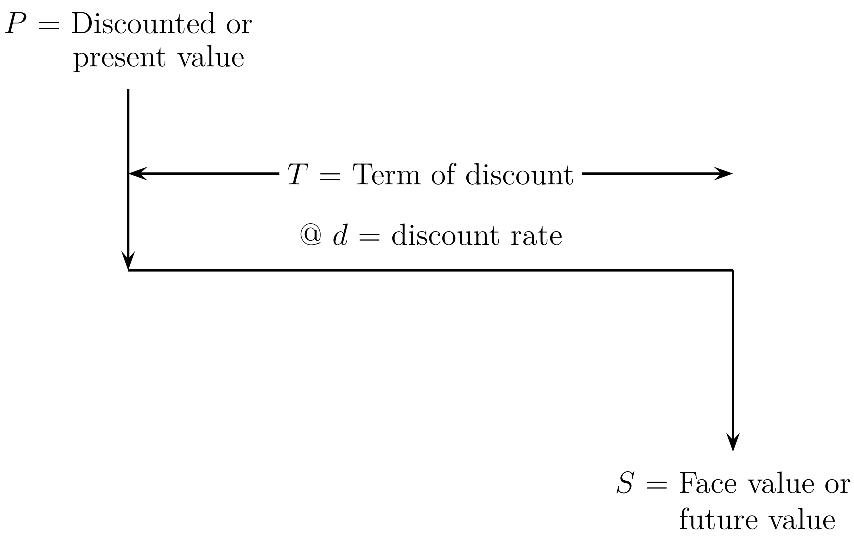
The discount D is given by

$$D = SdT$$

where d is the simple discount rate. The discounted (or present) value of S is

$$\begin{aligned} P &= S - D \\ &= S - SdT \\ &= S \times (1 - dT). \end{aligned}$$

This may be expressed in the form of the following time line:



Note two important aspects here. Firstly, this is structurally very similar to the time line for simple interest. However, secondly, since we are working with simple **discount**, and the discount rate is now expressed as a percentage of the future value (and not as a percentage of the present value as before), a negative sign appears in the formula. This means that the discount,

$$S \times d \times T,$$

is **subtracted** from the future value to obtain the present value.

Activity

Determine the simple discount on a loan of R3 000 due in eight months at a discount rate of 11% per annum. What is the discounted value of the loan? What is the equivalent simple interest rate R?

Answer

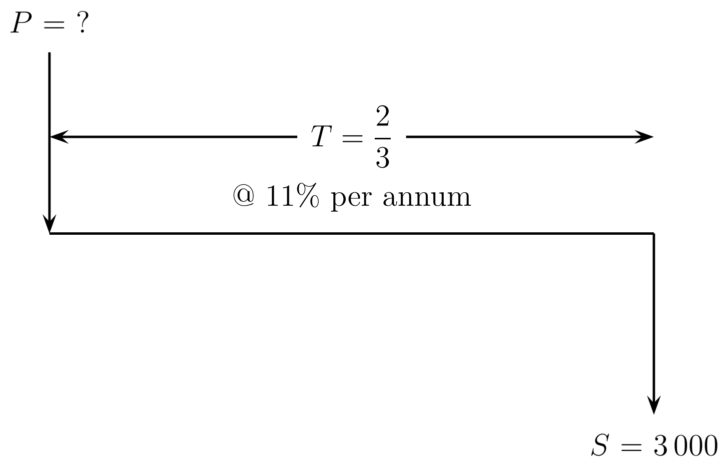
It is given that:

$S = 3\,000$

$d = 0{,}11$

$T = 8 \text{ months} = \frac{8}{12} = \frac{2}{3} \text{ year}$

This may be expressed in the form of the following time line:



The discount is calculated as

$$\begin{aligned} D &= SdT \\ &= 3\,000 \times 0{,}11 \times \frac{2}{3} \\ &= 220. \end{aligned}$$

That is, the simple discount is R220.

The discounted value is calculated as

$$\begin{aligned} P &= 3\,000 - 220 \\ &= 2\,780. \end{aligned}$$

The discounted value is R2 780.

In order to determine the equivalent interest rate R, we note that R2 780 is effectively the price now and that R3 000 is paid back eight months later. The interest is thus

$$\begin{aligned} I &= 3\,000 - 2\,780 \\ &= 220. \end{aligned}$$

The question can thus be restated. What simple interest rate, when applied to a principal of R2 780, will yield R220 interest in eight months? In this case

$$I = PRT$$

that is

$$\begin{aligned} 220 &= 2\,780 \times R \times \frac{2}{3} \\ R &= \frac{3}{2} \times \frac{220}{2\,780} \\ &= 0{,}1187. \end{aligned}$$

Thus the equivalent simple interest rate is 11,87% per annum.

Note the considerable difference between the interest rate of 11,87% and the discount rate of 11%. This emphasises the very important fact that the interest rate and the discount rate are not the same thing. The point is that they act on different amounts and at different times – the former acts on the present value; whereas the latter acts on the future value.

Exercise 7.2

1. A bank's simple discount rate is 10% per annum. If you take a loan and have to repay R4 000 in six months' time, how much would you receive from the bank now? What is the equivalent simple interest rate?
2. Determine the simple interest rate that is equivalent to a discount rate of
   (a) 12% for three months
   (b) 12% for nine months

   Hint: let $S = 100$ and use the appropriate formulae to set up an equation for R.
3. Mary needs R750 to buy a calculator. June is prepared to lend her the money on condition Mary repays her in ten months' time. Calculate the future value of the loan if June charges a 16% discount rate.

Study unit 7.2 Compound interest

Learning objectives: *On completion of this study unit you should be able to calculate compound interest.*

Suppose we invest R600 for three years at a simple interest rate of 10% per annum.

Easy, you say – at the end I will have earned

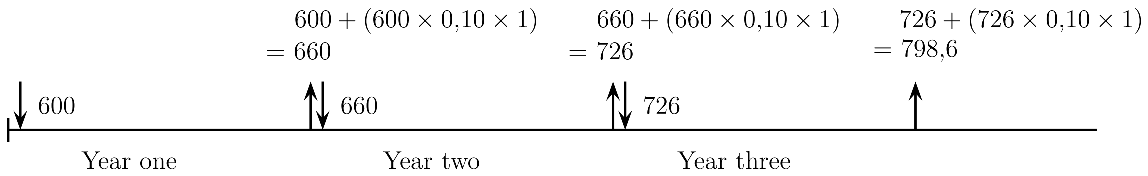
$$\begin{aligned} I &= PRT \\ &= 600 \times 0{,}10 \times 3 \\ &= 180 \end{aligned}$$

that is, R180 in interest and the accrued principal will be

$$600 + 180 = 780.$$

So actually to calculate the interest, you started every year with R600.

Now suppose that, instead of withdrawing the interest earned at the end of each year, you leave it in the account.



It sounds a better proposition to leave the interest in the bank and earn "interest on interest" or, in other words, "compound the interest". This is the basis for compound interest. Compound interest is just the repeated application of simple interest to an amount which is at each stage increased by the simple interest earned in the previous period. However, as the investment term stretches over a long period, we will have to do a lot of calculations. We therefore use the formula for compound interest instead, that is

$$S = P(1 + R)^T$$

where

S	is the **accrued principal** (total money at end of period)
P	is the **principal** (original amount invested or borrowed)
R	is the **rate of interest** per period, given as a decimal fraction
T	is the **number of periods** of investment

Consider the following example:

Calculate the compounded amount on R500 invested for 10 years at $7\frac{1}{2}\%$ interest per annum and compounded annually.

Remember to write down all information given:

principal = R500, that is $P = 500$

term = 10 years, that is $T = 10$

interest rate = $7\frac{1}{2}\%$ compound annually, that is $R = 0{,}075$

Now we have

$$\begin{aligned} S &= P(1+R)^T \\ &= 500(1+0{,}075)^{10} \\ &= 500 \times 1{,}075^{10} \\ &= 1\,030{,}52. \end{aligned}$$

The compounded amount is R1 030,52.

(*See Tutorial Letter 101: Using the recommended calculator.*)

If you do not have a "power to the x" key on your calculator (on a SHARP calculator this is the $[y^x]$ key), you have to do the following:

$$1{,}075 \times 1{,}075 \times 1{,}075 \ \ldots \ \times 1{,}075 \quad \text{ten times.}$$

What if interest is compounded quarterly?

Say we invest R1 000 for one year at 8% per annum compounded quarterly. Now we have:

$P = 1\,000$

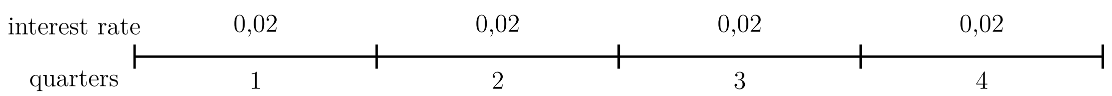
$R = 8\% = 0{,}08$ per annum

$T = 1$ year $= 4$ quarters

But the interest is compounded quarterly, and there are four quarters in a year. We cannot just make T equal to 4 – what about the interest rate? The interest rate used per quarter is

$$R = \frac{0{,}08}{4} = 0{,}02.$$

The interest rate is represented on a time line as follows:



$R = 0{,}08$ per annum or 0,02 per quarter
one year = 4 quarters

We now have

$$\begin{aligned} S &= P(1+R)^T \\ &= 1\,000(1+0{,}02)^4 \\ &= 1\,000 \times 1{,}02^4 \\ &= 1\,082{,}43. \end{aligned}$$

The compounded amount is R1 082,43.

(*See Tutorial Letter 101: Using the recommended calculator.*)

Another way of doing the same thing is

$$S = 1\,000\left(1 + \frac{0{,}08}{4}\right)^{1\times 4}.$$

Exercise 7.3

1. Determine the interest earned if R1 000 is invested for one year at 8% per annum. But the interest is compounded biannually (that is every six months).
2. What is the total amount available after $2\frac{1}{2}$ years if R2 000 is invested at an interest rate of 12% per annum compounded quarterly?
3. You wish to invest R1 000 for two years. Which of the following investment opportunities will give you the best return on money invested?
   (a) 10% simple interest per annum
   (b) $9\frac{1}{2}$% interest per annum compounded biannually
   (c) 9% interest per annum compounded quarterly

Study unit 7.3 The time value of money

Learning objectives: *On completion of this study unit you should*

- *give and apply the equations and the corresponding time lines relating present and future values of money when compounding is applicable*
- *state and apply the two rules for moving money backwards and forwards in time*
- *use the rules to replace one set of financial obligations with another, that is, to reschedule debts*

From time to time, a debtor (the person who owes money) may wish to replace his set of financial obligations with another set. On such occasions, he must negotiate with his creditor (the person who is owed money) and agree on a new due date, as well as a new interest rate. This is generally achieved by evaluating each obligation in terms of the new due date, and equating the sum of the old and the new obligations on the new date. The resultant equation of value is then solved to obtain the new future value which must be repaid on the new due date.

It is evident from these remarks that time value of money concepts must play an important role in any such considerations.

We shall first review the present and future value concepts.

You will recall that if P is invested at an interest rate of R per period for a term of length T periods, it accumulates to

$$S = P(1+R)^T$$

and we call P the present value and S the future value of the investment.

Sometimes, we are given the future value S and wish to know the present value P. To do this, we can rearrange the above formula as follows:

$$\begin{aligned} P &= S \div (1+R)^T \\ &= S(1+R)^{-T}. \end{aligned}$$

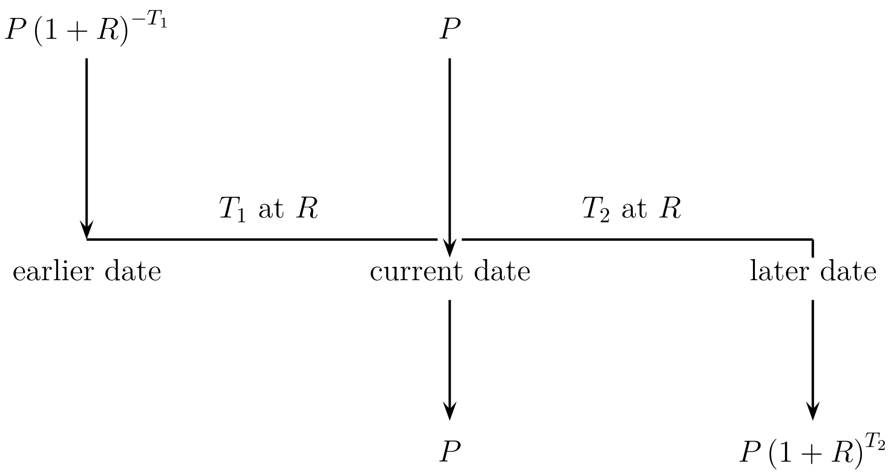
In the latter case, that is, finding the present value of some future value, PV is often referred to as the discounted value of the future sum, and the factor $(1+R)^{-T}$ is termed the discount factor. This should not be confused with the simple discounting used in study unit 4.1, and for which a discount rate was defined. The rate used in compound discounting is simply the relevant interest rate. It would indeed be possible to introduce a compound discount rate, but this is not used in practice since it does not have a readily understood meaning. The important concepts are those of present and future value, and the fact that the process of determining present value is often referred to as discounting.

For example: You decide now that when you graduate in four years' time, you are going to treat yourself to a car to the value of R120 000. If you earn 11,5% interest compounded monthly on an investment, calculate the amount that you need to invest now.

The money needed now is

$$P = \frac{S}{(1+R)^T} = \frac{120\,000}{\left(1+\frac{0{,}115}{12}\right)^{48}} = 75\,920{,}13.$$

You need to invest R75 920,13 now. We can summarise the above formulae succinctly by means of a time line.



We can formulate the above results as two simple rules:

1. To move money **forwards** (determine a future value), inflate the relevant sum by **multiplying** by the **accumulation** factor $(1+R)^T$.

2. To move money **backwards** (determine a present value), deflate the relevant sum by **dividing** by the **accumulation** factor $(1+R)^T$.

Note: The second rule is equivalent to multiplying by a **discount factor** of $(1+R)^{-T}$.

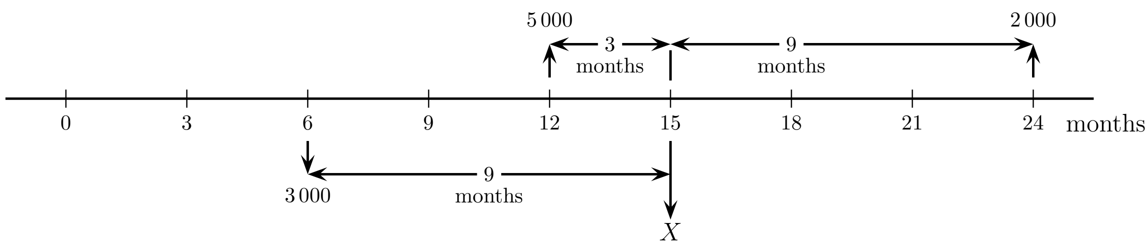
The point is that the mathematics of finance deals with dated values of money. This fact is fundamental to any financial transaction involving money due on different dates. In principle, every sum of money specified should have an attached date. Fortunately, in practice it is often clear from the context what the implied date is.

Also note that, in time value of money, payments equal obligations as illustrated in the following example:

Mr More Money owes R5 000 due in one year's time and R2 000 due in two years' time. Money is worth 12% per annum, compounded monthly.

He decides to reschedule his payments by paying R3 000 six months from now and the rest fifteen months from now. What will his last payment fifteen months from now be?

First we represent the money on the time line:



@ 12% per annum compounded monthly

Secondly, we determine each payment on the new due date of 15 months hence.

Obligations: There are two obligations, namely R5 000 and R2 000.

For the R5 000 obligation:

We must move this R5 000 from month 12 (when it was originally due) to the new due date of 15 months. Thus T is equal to three months ($T = 15 - 12 = 3$). The future value is

$$\begin{aligned} S &= P(1+R)^T \\ &= 5\,000\left(1+\frac{0{,}12}{12}\right)^3 \\ &= 5\,151{,}51. \end{aligned}$$

The R5 000 obligation is worth R5 151,51 at month 15.

For the R2 000 obligation:

We are discounting the R2 000 back (from month 24 to month 15) nine times. Thus T is equal to nine months ($T = 24 - 15 = 9$). The present value is calculated as

$$\begin{aligned} S &= P(1+R)^T \\ 2\,000 &= P\left(1+\frac{0{,}12}{12}\right)^9 \\ P &= \frac{2\,000}{\left(1+\frac{0{,}12}{12}\right)^9} \\ &= 2\,000\left(1+\frac{0{,}12}{12}\right)^{-9} \\ &= 1\,828{,}68. \end{aligned}$$

The R2 000 obligation is worth R1 828,68 at month 15.

Payments: There are two payments, namely R3 000 and Rx.

For the R3 000 payment:

We move the R3 000 to the new date. Then T is equal to nine months ($T = 15 - 6 = 9$). The future value is

$$\begin{aligned} S &= P(1+R)^T \\ &= 3\,000\left(1+\frac{0{,}12}{12}\right)^9 \\ &= 3\,281{,}06. \end{aligned}$$

The R3 000 payment is worth R3 281,06 at month 15.

For the Rx payment:

As this is the last payment no interest is involved and the payment remains Rx.

Remember the total amount to be paid is equal to the total obligations. Therefore

$$\begin{aligned}
\text{payments} &= \text{obligations} \\
3\,000\left(1+\frac{0{,}12}{12}\right)^{9}+x &= 5\,000\left(1+\frac{0{,}12}{12}\right)^{3}+2\,000\left(1+\frac{0{,}12}{12}\right)^{-9} \\
3\,281{,}06+x &= 5\,151{,}51+1\,828{,}68 \\
3\,281{,}06+x &= 6\,980{,}19 \\
x &= 6\,980{,}19-3\,281{,}06 \\
&= 3\,699{,}13.
\end{aligned}$$

Therefore he will pay R3 699,13 to settle his account at the end of fifteen months.

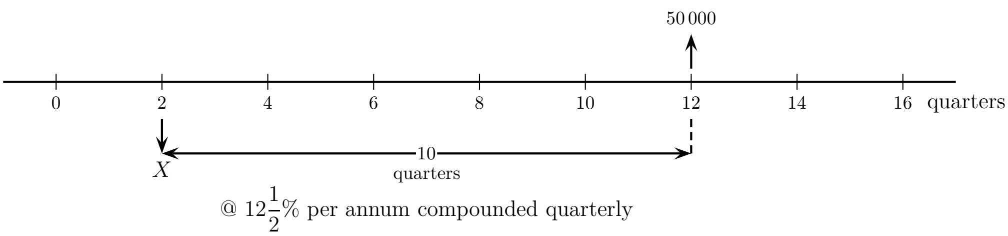
Activity

An obligation of R50 000 falls due in three years. The interest is credited quarterly at an interest rate of 12,5% per annum. What amount will cover the debt if it is paid as follows:

1. in six months
2. in four years

Answer

1. Draw the relevant time line.



To determine the debt if it is paid in six months (ie two quarters from now), we must discount the debt back two and a half years (ie $2{,}5 \times 4 = 10$ quarters) from the due date to obtain the amount due.

We have that:

$S = 50\,000$

$R = \dfrac{0{,}125}{4}$

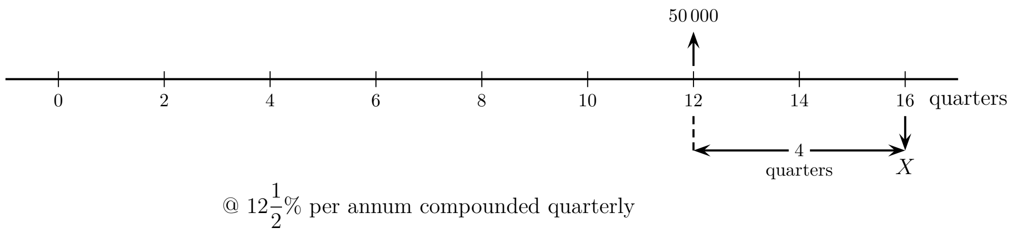
$T = 10$ quarters

The present value is calculated as

$$\begin{aligned}
P &= S\,(1+R)^{-T} \\
&= 50\,000 \times \left(1+\frac{0{,}125}{4}\right)^{-10} \\
&= 36\,756{,}18.
\end{aligned}$$

The amount due after six months is R36 756,18.

2. Draw the relevant time line.



To determine the debt if it is allowed to accumulate until one year after the due date, we must move the money forward one year (four quarters) to obtain

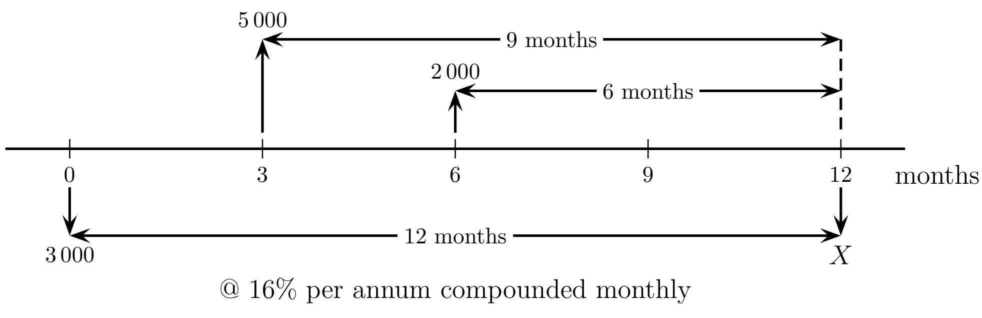
$$\begin{aligned} S &= 50\,000 \times \left(1 + \frac{0{,}125}{4}\right)^4 \\ &= 56\,549{,}12. \end{aligned}$$

The amount due after four years is R56 549,12.

I said above that we would concern ourselves here with the replacement of one set of financial obligations with another equivalent set. This sounds complicated, but is really just a case of applying the above rules for moving money backwards and forwards, of keeping a clear head and remembering that, at all times, the only money that may be added together (or subtracted) are those with a common date. An example will illustrate the process.

Maxwell Moneyless owes Bernard Broker money. An amount of R5 000 is due in three months from now, and R2 000 is due in six months from now. Maxwell offers to pay R3 000 immediately if he can pay the balance in one year's time. Bernard agrees, on condition that they use an interest rate of 16% per annum, compounded monthly. They also agree that, for settlement purposes, the R3 000 paid now will also be subject to the same rate. How much will Maxwell have to pay at the end of the year?

All amounts are shown on the following time line, with debts above the line and payments below.



Debts: The numerical values of the R5 000 and R2 000 debts must be moved to 12 months.

For the R5 000 debt:

The future value is

$$\begin{aligned} S &= 5\,000 \times \left(1 + \frac{0{,}16}{12}\right)^9 \\ &= 5\,633{,}02. \end{aligned}$$

The amount due at the end of the year is R5 633,02.

For the R2 000 debt:

The future value is

$$\begin{aligned} S &= 2\,000 \times \left(1 + \frac{0{,}16}{12}\right)^{6} \\ &= 2\,165{,}43. \end{aligned}$$

The amount due at the end of the year is R2 165,43.

Payments: There are two payments, namely R3 000 at month zero and Rx at month twelve.

For the R3 000 payment:

The future value is

$$\begin{aligned} S &= 3\,000 \times \left(1 + \frac{0{,}16}{12}\right)^{12} \\ &= 3\,516{,}81. \end{aligned}$$

The value of the R3 000 at the end of the year is R3 516,81.

For the Rx payment:

As this is the last payment no interest is involved and the payment remains Rx.

Remember the total amount to be paid is equal to the total debts. Therefore

$$\begin{aligned} \text{payments} &= \text{debts} \\ 3\,516{,}81 + x &= 5\,633{,}02 + 2\,165{,}43 \\ 3\,516{,}81 + x &= 7\,798{,}45 \\ x &= 7\,798{,}45 - 3\,516{,}81 \\ &= 4\,281{,}64. \end{aligned}$$

That is, under the agreement, Maxwell will have to pay Bernard R4 281,64 in one year's time.

Exercise 7.4

1. Melanie owes R500 due in eight months. The interest rate is 15% per annum, compounded monthly. What single payment will repay her debt in the following cases?
   (a) now
   (b) six months from now
   (c) in one year's time
2. Mary-Jane must pay the bank R2 000 which is due in one year (interest is included). She is anxious to lessen her burden in advance and therefore pays R600 after three months, and another R800 three months later. If the bank agrees that both payments are subject to the same interest rate as the loan namely 12% per annum compounded quarterly, how much must she pay at the end of the year to settle her outstanding debt?

Study unit 7.4 Annuities

Learning objectives: *On completion of this study unit you should*

- *explain the basic structure and elements of an annuity*
- *determine the future value of an annuity*
- *determine the present value of an annuity*

An **annuity** is a sequence of **equal** payments at **equal** intervals of time.

Examples of annuities include the annual premium payable on a life insurance policy, interest payments on a bond, quarterly stock dividends, and so on.

The **payment interval** (or **period**) of an annuity is the time between successive payments, while the **term** is the time from the beginning of the first payment interval to the end of the last payment interval.

An annuity is termed an **ordinary** annuity when payments are made at the same time as interest is credited, that is, at the **end** of the payment intervals. By contrast, an annuity whose periodic payment is made at the **beginning** of each payment interval is known as an **annuity due**.

If the payments begin and end on a fixed date, the annuity is known as an **annuity certain**. On the other hand, if the payments continue forever, the annuity is known as **perpetuity**.

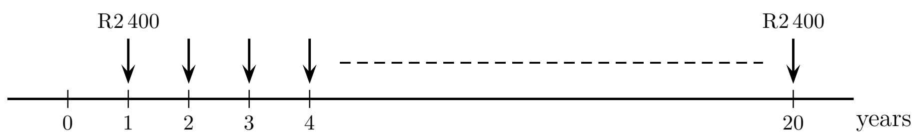
The **amount** or **future value** of an annuity is the sum of all payments made and of all accumulated interest at the end of the term.

The **present value** is the sum of all payments, each discounted to the beginning of the term, that is, the sum of the present values of all payments.

The basic concepts are illustrated by the following examples (notice how time lines, with the interest period as the unit, are used).

Example 1

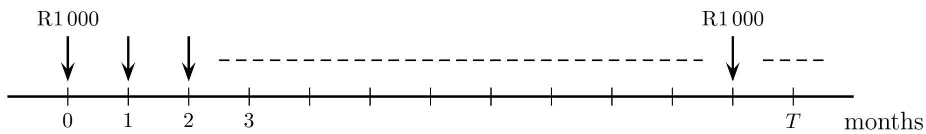
The premium on an endowment policy is R2 400 per year payable for 20 years, and payable at the end of each period.



Here, the payment interval is one year and the term is 20 years. Since the payments are made at the end of each period, this is an ordinary annuity certain.

Example 2

The monthly rent for a shop is R1 000, payable in advance.



Here, the payment interval is one month, while the term will be determined by the contract. Since the payment is made at the beginning of each period, this is an annuity due.

Example 3

A company is expected to pay R180 indefinitely every six months on a share of its preferred stock.

This is an example of a perpetuity with a payment interval of six months. In this case, no term is defined, since payments continue indefinitely.

In this study unit we shall concentrate mainly on the ordinary annuity certain. From now on, we shall simply speak of "an annuity", which, unless otherwise stated, will mean an ordinary annuity certain.

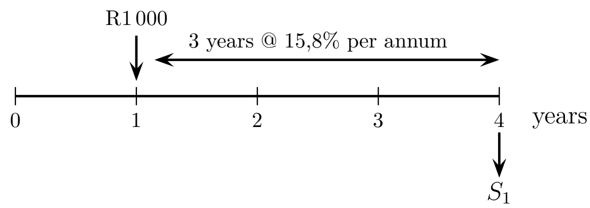
We first look at an activity in which the amount of an annuity is calculated, and then derive a general formula for the amount.

Activity

Determine the accumulated amount of an annuity after four payments, each of R1 000 paid annually, and at an interest rate of 15,8% per annum.

Answer

At the end of the term, the first payment of R1 000 will have accumulated interest for 3 years, compounded at 15,8% per annum as indicated by the following time line.



Its accumulated value, S_1, at the end of the term is

$$\begin{aligned} S_1 &= 1\,000\,(1+0{,}158)^3 \\ &= 1\,552{,}84. \end{aligned}$$

The accumulated value is R1 552,84.

This is obviously a straightforward application of the compound interest formula.

Similarly, the accumulated values of the second, third and fourth payments at the end of the term are respectively

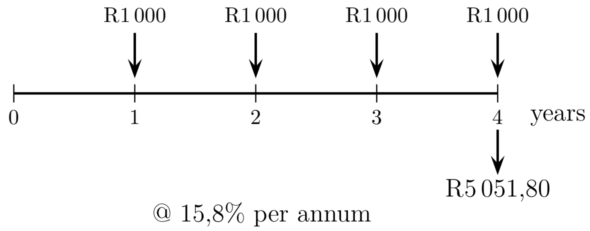
$$\begin{aligned} S_2 &= 1\,000\,(1+0{,}158)^2 \\ &= 1\,340{,}96 \\ S_3 &= 1\,000\,(1+0{,}158)^1 \\ &= 1\,158{,}00 \\ S_4 &= 1\,000 \end{aligned}$$

The accumulated amount of the annuity after four payments is

$$\begin{aligned} S &= S_1+S_2+S_3+S_4 \\ &= 1\,000\,(1+0{,}158)^3+1\,000\,(1+0{,}158)^2+1\,000\,(1+0{,}158)+1\,000 \\ &= 5\,051{,}80. \end{aligned}$$

The accumulated amount is thus R5 051,80.

The result is represented by the following time line.



(*See Tutorial Letter 101: Using the recommended calculator.*)

In study units 4.1 to 4.3, the number of periods and the interest rate were represented by T and R respectively. Now these are represented by n and i respectively. The letter R is now used to represent the payment.

If the payment in rand made at each payment interval in respect of an ordinary annuity certain, at interest rate i per payment interval, is R, then the **amount** or **future value** of the annuity after n intervals is

$$S = R\left[\frac{(1+i)^n - 1}{i}\right].$$

The measure $\frac{(1+i)^n - 1}{i}$ is usually denoted by $s_{\overline{n}|i}$. Thus

$$S = Rs_{\overline{n}|i}.$$

Next, we look at the present value of an annuity. This is the sum of the present values of all payments.

The following example should clarify the concepts involved.

Calculate the present value of an annuity, which provides R1 000 per year for five years if the interest rate is 12,5%. The present value of the first payment is P_1:

$$\begin{aligned} P_1 &= 1\,000 \div (1+0{,}125) \\ &= 888{,}89. \end{aligned}$$

Remember this means that R888,89 invested now at $12\frac{1}{2}\%$ will yield R1 000 in one year's time.

Similarly, the present values of the other four payments are

$$\begin{aligned} P_2 &= 1\,000 \div (1+0{,}125)^2 \\ &= 790{,}12 \\ P_3 &= 1\,000 \div (1+0{,}125)^3 \\ &= 702{,}33 \\ P_4 &= 1\,000 \div (1+0{,}125)^4 \\ &= 624{,}30 \\ P_5 &= 1\,000 \div (1+0{,}125)^5 \\ &= 554{,}93 \end{aligned}$$

Thus, the present value of the annuity is

$$\begin{aligned} P &= P_1 + P_2 + P_3 + P_4 + P_5 \\ &= 1\,000\,(1{,}125)^{-1} + 1\,000\,(1{,}125)^{-2} + 1\,000\,(1{,}125)^{-3} + 1\,000\,(1{,}125)^{-4} + 1\,000\,(1{,}125)^{-5} \\ &= 3\,560{,}57. \end{aligned}$$

Thus, R3 560,57 must be invested now at $12\frac{1}{2}\%$ to provide for five payments of R1 000 each at yearly intervals, commencing one year from today.

(*See Tutorial Letter 101: Using the recommended calculator.*)

This calculation was performed for a periodic payment of R1 000. If the payment is R1 per period, then the present value of the amount S, which has accumulated by the end of the n^{th} period, is

$$P = S \div (1+i)^n,$$

but

$$S = \frac{(1+i)^n - 1}{i} \quad \text{(see previous page)}$$

so

$$P = \frac{(1+i)^n - 1}{i(1+i)^n}.$$

This is usually denoted by $a_{\overline{n}|i}$. If the payment in rand made at each payment interval for an ordinary annuity certain, at interest rate i per payment interval, is R for a total of n payments, then the **present value** is

$$\begin{aligned} P &= Ra_{\overline{n}|i} \\ &= R\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right]. \end{aligned}$$

Note: The relationship between $a_{\overline{n}|i}$ and $s_{\overline{n}|i}$ is simply

$$s_{\overline{n}|i} = (1+i)^n a_{\overline{n}|i}.$$

Thus $s_{\overline{n}|i}$ is the future value of $a_{\overline{n}|i}$, and $a_{\overline{n}|i}$ is the present value of $s_{\overline{n}|i}$.

Exercise 7.5

1. Determine the accumulated amount of an annuity after five payments of R600, each paid annually, at an interest rate of 10% per annum.
2. Mrs Dooley decides to save for her daughter's higher education, and every year, from the child's first birthday onwards, puts away R1 200. If she receives 11% interest, what will the accumulated amount be after her daughter's eighteenth birthday?
3. What is the accumulated amount of an annuity with a payment of R600 four times per year, and an interest rate of 13% per annum compounded quarterly, at the end of a term of five years?
4. Max pays R1 000 down on a second-hand motorbike and contracts to pay the balance in 24 monthly instalments of R200. If interest is charged at a rate of 12% per annum, payable monthly, how much did the motorbike originally cost when Max purchased it? How much interest does he pay?
5. Determine the present value of an annuity with half-yearly payments of R800 compounded half-yearly at an interest rate of 12,5% per annum, and with a term of ten years.

Study unit 7.5 Amortisation

Learning objectives: *On completion of this study unit you should*

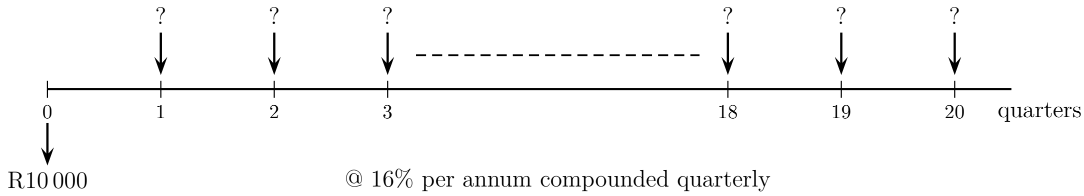
- *calculate the payments on a mortgage loan*
- *set up an amortisation schedule for a loan*

A loan is said to be **amortised** when all liabilities (ie both principal and interest) are paid by a sequence of equal payments made at equal intervals of time.

Example 1

A loan of R10 000 with interest of 16% compounded quarterly is to be amortised by equal quarterly payments over five years. The first payment is due at the end of the first quarter. Calculate the payment.

The representation time line is as follows:



It is evident that the 20 payments form an ordinary annuity with a present value of R10 000 and an interest rate of 16% ÷ 4 = 4%. It is thus given that:

$P = 10\,000$

$i = \dfrac{0{,}16}{4} = 0{,}04$ per quarter

$n = 5 \times 4 = 20$ quarters.

Thus

$$\begin{aligned} 10\,000 &= R\,a_{\overline{20}|0{,}04} \\ R &= 10\,000 \div \left[\frac{(1+i)^n - 1}{i\,(1+i)^n}\right] \\ R &= 10\,000 \div \left[\frac{1{,}04^{20} - 1}{0{,}04 \times 1{,}04^{20}}\right] \\ &= 735{,}82. \end{aligned}$$

Thus, the quarterly payment is R735,82.

(*You may also use your calculator to determine this value. See Tutorial Letter 101.*)

This example illustrates the typical amortisation problem. **A loan of present value P rand must be amortised over n payments at interest rate i per payment interval.** What is the amount of the payment R? As pointed out in the example, the n payments form an ordinary annuity; hence, we can use the formula for the present value of such an annuity to calculate R, namely

$$P = R\,a_{\overline{n}|i}$$

or

$$R = P \div a_{\overline{n|}i}.$$

If a loan of present value P rand must be amortised over n payments at interest rate i per payment interval then the amount of the payment R is

$$R = P \div a_{\overline{n|}i}.$$

Let us pause for a moment to think about the mechanics of amortisation. Initially, the total amount loaned (ie the present value at instant 0) is owed. However, as payments are made, the **outstanding principal**, or **outstanding liability** as it is also known, decreases until it is eventually zero at the end of the term. At the end of each payment interval, the interest on the outstanding principal is first calculated. The payment R is then first used to pay the interest due. The balance of the payment is thereafter used to reduce the outstanding principal. (If, for some reason or another, for example by default, no payment is made, the interest owed is added to the outstanding principal and the outstanding debt then increases. Usually, this will be accompanied by a rather strongly worded letter of warning to the debtor!) Since the outstanding principal decreases with time, the interest owed at the end of each period also decreases with time. This means that the fraction of the payment which is available for reducing the principal increases with time.

Note that, at any stage of the term, the amount outstanding just after a payment has been made is the present value of all payments that still have to be made.

The above concepts are all embodied and illustrated in the **amortisation schedule**, which is a table indicating the distribution of each payment in regard to interest and principal reduction. This is illustrated in the following example.

Draw up an amortisation schedule for a loan of R5 000 which is repaid in annual payments over five years at an interest rate of 15% per annum.

(a) It is given that:

$P = 5\,000$

$i = 0{,}15$ per year

$n = 5$ years

The payments are

$$\begin{aligned} P &= R\,a_{\overline{n|}i} \\ 5\,000 &= R\,a_{\overline{5|}0{,}15} \\ R &= 1\,491{,}58, \end{aligned}$$

or

$$\begin{aligned} P &= R\left[\frac{(1+i)^n - 1}{i\,(1+i)^n}\right] \\ 5\,000 &= R\left[\frac{1{,}15^5 - 1}{0{,}15 \times 1{,}15^5}\right] \\ R &= \frac{5\,000 \times 0{,}15 \times 1{,}15^5}{1{,}15^5 - 1} \\ &= 1\,491{,}58. \end{aligned}$$

Therefore, the payments are R1 491,58 per year for 5 years.

(b) The interest due at the end of the first year is

$$\begin{aligned} I &= PRT \\ &= 5\,000 \times 0{,}15 \times 1 \\ &= 750. \end{aligned}$$

Therefore, the interest due at the end of the first year is R750. Note that you have to keep T equal to 1 when you calculate interest at the end of the second to the fifth year. P is the outstanding amount at the start of the year.

(c) The principal repaid at the end of the first year is

$$\begin{aligned} \text{principal repaid} &= \text{payment} - \text{interest due} \\ &= 1\,491{,}58 - 750 \\ &= 741{,}58. \end{aligned}$$

Therefore, the principal repaid at the end of the first year is R741,58.

(d) The outstanding principal at the beginning of the second year is

$$\begin{aligned} &= \text{outstanding principal from previous year} - \text{principal repaid from previous year} \\ &= 5\,000 - 741{,}58 \\ &= 4\,258{,}42. \end{aligned}$$

Therefore the outstanding principal at the beginning of the second year is R4 258,42.

The amortisation table is given below.

Year	(d) Outstanding principal at year beginning	(b) Interest due at year end (simple)	(a) Payment	(c) Principal repaid
1	5 000,00	750,00	1 491,58	741,58
2	4 258,42	638,76	1 491,58	852,82
3	3 405,60	510,84	1 491,58	980,74
4	2 424,86	363,73	1 491,58	1 127,85
5	1 297,01	194,55	1 491,58	1 297,03
Total		2 457,88	7 457,90	5 000,02

(*However, it is easier to use the recommended calculator. See Tutorial Letter 101: Using the recommended calculator.*)

Note:

1. The interest due at the end of each year is simply 15% of the outstanding principal.
2. The principal repaid is the difference between the payment and the interest due.
3. The outstanding principal at the beginning of the year is equal to the outstanding principal at the beginning of the previous year minus the principal repaid of the previous year.
4. Also note that, due to rounding errors, the total principal repaid is in error by two cents.

As stated above, and as is well illustrated in the above amortisation schedule, the outstanding principal decreases over the term from the amount initially borrowed to zero at the end.

You may well argue that it is all very well doing such calculations but everyone knows that, nowadays, interest rates on mortgage bonds do not hold for very long. So what is the point of talking about equal payments, and, if the interest rate does change, how can we handle it?

Yes, it is true that interest rates do change from time to time. But, the implication is that the payments also change accordingly, as you may possibly have experienced to your irritation if you have a house with a mortgage bond on it. To recalculate the payments, we simply work from the present value, which is the outstanding principal on the date from which the change is implemented. Usually, it is assumed that the number of payments still due remains the same, although, in some cases, the term of the loan can be extended.

The following example illustrates a calculation of this type:

Jonathan purchases an apartment by making a down payment of R40 000, and obtains a 20-year loan for the balance of R280 000 at 15% per annum, compounded monthly. After $4\frac{1}{2}$ years, the bank adjusts the interest rate to 16%. What is the new amount which he must pay if the term of the loan remains the same?

Initially, the interest rate was $15\% \div 12 = 1{,}25\%$ per month, and the number of payments to be made was $20 \times 12 = 240$. Thus, initially his payments were

$$\begin{aligned} R &= 280\,000 \div a_{\overline{240}|0,0125} \\ &= 3\,687{,}01. \end{aligned}$$

His initial payments were R3 687,01.

After $4\frac{1}{2}$ years, that is, after 54 payments $\left(4\frac{1}{2} \times 12\right)$ the present value of the loan is

$$\begin{aligned} P &= 3\,687{,}01\, a_{\overline{186}|0,0125} \\ &= 265\,699{,}85. \end{aligned}$$

After $4\frac{1}{2}$ years the present value of the loan is R265 699,85. The value of n is now calculated as

$$240 - 54 = 186.$$

But now the outstanding principal of R265 699,85 must be amortised over the remaining 186 periods (ie $15\frac{1}{2}$ years) at an interest rate of $\frac{16}{12}\% = \frac{4}{3}\%$ per month. Thus, the new payments are now

$$\begin{aligned} R &= 265\,699{,}85 \div a_{\overline{186}|\frac{4}{3}\div 100} \\ &= 3\,872{,}30. \end{aligned}$$

The new payments are now R3 872,30.

Exercise 7.6

1. You purchase a house for R270 000 with a down payment (often referred to as a deposit) of R45 000. You secure a mortgage bond with a building society for the balance at 11,5% per annum compounded monthly, with a term of 20 years. What are the monthly payments?
2. Your Great-Aunt Agatha dies and leaves you an inheritance of R60 000, which is to be paid to you in ten annual payments, at the end of each year. If the money is invested at 12% per annum, how much do you receive each year?
3. Draw up an amortisation schedule for a loan of R4 000 for three years at 15% per annum compounded half-yearly and repayable in six half-year payments.

APPENDIX A

Answers to exercises

Component 1. Numbers and working with numbers

Exercise 1.1:1
Applying the distributive law of multiplication over addition gives

$$\begin{aligned} & 7 \times (6 \times (5+4) + 3 \times (2+1)) \\ = & \ 7 \times (6 \times 5 + 6 \times 4 + 3 \times 2 + 3 \times 1) \\ = & \ 7 \times 6 \times 5 + 7 \times 6 \times 4 + 7 \times 3 \times 2 + 7 \times 3 \times 1. \end{aligned}$$

Exercise 1.1:2(a)
The associative law of addition is associated with

$$2 + (5+4) = (2+5) + 4.$$

Exercise 1.1:2(b)
The commutative law of addition is associated with

$$(3+7) + 4 = (7+3) + 4.$$

Exercise 1.1:2(c)
The commutative law of multiplication is associated with

$$(7 \times 5) \times 2 = 2 \times (7 \times 5).$$

Exercise 1.1:2(d)
The associative law of multiplication is associated with

$$2 \times (7 \times 4) = (4 \times 2) \times 7.$$

Exercise 1.2:1(a)
Substituting x with 2 gives

$$\begin{aligned} 12 \times 2 + 17 &= 24 + 17 \\ &= 41. \end{aligned}$$

Exercise 1.2:1(b)
Substituting x with 4 gives

$$\begin{aligned} 4^2 - 3 &= 16 - 3 \\ &= 13. \end{aligned}$$

Exercise 1.2:1(c)
Substituting x with 3 gives

$$\begin{aligned} 2 \times 3^2 &= 2 \times 9 \\ &= 18. \end{aligned}$$

Exercise 1.2:1(d)
Substituting x with 5 gives

$$\begin{aligned} 4 \times 5 - 1 &= 20 - 1 \\ &= 19. \end{aligned}$$

Exercise 1.2:1(e)
Substituting x with 5 gives

$$\begin{aligned} \frac{5+7}{4} + 3 &= \frac{12}{4} + 3 \\ &= 3 + 3 \\ &= 6. \end{aligned}$$

Exercise 1.2:1(f)
Substituting x with 6 gives

$$\begin{aligned} (6+3)(6-2) &= 9 \times 4 \\ &= 36. \end{aligned}$$

Exercise 1.2:1(g)
Substituting x with 1 gives

$$\begin{aligned} 10 - 3 \times 1 &= 10 - 3 \\ &= 7. \end{aligned}$$

Exercise 1.2:1(h)
Substituting x with 12 gives

$$\begin{aligned} \frac{12}{2} + \frac{12}{3} &= 6 + 4 \\ &= 10. \end{aligned}$$

Exercise 1.2:2(a)
Substituting x with 3 gives

$$\begin{aligned} 5 \times 3 + 7 &= 15 + 7 \\ &= 22. \end{aligned}$$

Exercise 1.2:2(b)
Substituting x with 3 gives

$$\begin{aligned} 3 + 3 \times 3 - 1 &= 3 + 9 - 1 \\ &= 11. \end{aligned}$$

Exercise 1.2:2(c)
Substituting x with 3 gives

$$\begin{aligned} 5 \times 3^2 - 9 &= 5 \times 9 - 9 \\ &= 45 - 9 \\ &= 36. \end{aligned}$$

Exercise 1.2:2(d)
Substituting x with 3 gives

$$\begin{aligned} \frac{3+4}{7} &= \frac{7}{7} \\ &= 1. \end{aligned}$$

Exercise 1.2:2(e)
Substituting x with 3 gives

$$\begin{aligned} 2\,(3+4) &= 2 \times 7 \\ &= 14. \end{aligned}$$

Exercise 1.2:2(f)
Substituting x with 3 gives

$$7 - 3 + 2 = 6.$$

Exercise 1.2:3(a)
Substituting a with 2, b with 1 and c with 7, gives

$$\begin{aligned} 2(2+1-7) + 7(1-2) &= 2(-4) + 7(-1) \\ &= 2 \times -4 + 7 \times -1 \\ &= -8 - 7 \\ &= -15. \end{aligned}$$

Exercise 1.2:3(b)
Substituting a with 2, b with 1 and c with 7, gives

$$\begin{aligned} 2^2 + 1^2 + 7^2 &= 4 + 1 + 49 \\ &= 54. \end{aligned}$$

Exercise 1.2:3(c)
Substituting a with 2, b with 1 and c with 7, gives

$$\begin{aligned} (2+1)(1-7) &= 3 \times -6 \\ &= -18. \end{aligned}$$

Exercise 1.2:3(d)
Substituting a with 2, b with 1 and c with 7, gives

$$\begin{aligned} 2 \times 1 - \frac{7+3}{2} + 1^2 &= 2 - \frac{10}{2} + 1 \\ &= 2 - 5 + 1 \\ &= -2. \end{aligned}$$

Exercise 1.2:4(a)
The mathematical expression is

$$x + y.$$

Exercise 1.2:4(b)
The mathematical expression is

$$8 - (a + b).$$

Exercise 1.2:4(c)
The mathematical expression is

$$3 \times x + 2 \times y = 3x + 2y.$$

Exercise 1.2:4(d)
The mathematical expression is

$$y + 7.$$

Exercise 1.3:1
Simplifying gives

$$8 \times \frac{4}{1} = 32.$$

Exercise 1.3:2
Simplifying gives

$$12 \times \frac{5}{1} = 60.$$

Exercise 1.3:3
Simplifying gives

$$\begin{aligned} \frac{8}{13} \times \frac{25}{12} &= \frac{200}{156} \\ &= \frac{50}{39} \\ &= 1\frac{11}{39}. \end{aligned}$$

Exercise 1.3:4
Simplifying gives

$$\begin{aligned} \frac{\frac{7}{2}}{7} &= \frac{7}{2} \times \frac{1}{7} \\ &= \frac{1}{2}. \end{aligned}$$

Exercise 1.3:5
Simplifying gives

$$\begin{aligned} \frac{\frac{33}{4}}{\frac{1}{2}} &= \frac{33}{4} \times \frac{2}{1} \\ &= \frac{66}{4} \\ &= \frac{33}{2} \\ &= 16\frac{1}{2}. \end{aligned}$$

Exercise 1.3:6
Simplifying gives

$$\begin{aligned}\frac{17}{20}+\frac{1}{4}-\frac{3}{5} &= \frac{17+1\times 5-3\times 4}{20}\\ &= \frac{17+5-12}{20}\\ &= \frac{10}{20}\\ &= \frac{1}{2}.\end{aligned}$$

Exercise 1.3:7
Simplifying gives

$$\begin{aligned}\frac{2}{3}+5-\frac{6}{7} &= \frac{2\times 7+5\times 21-6\times 3}{21}\\ &= \frac{14+105-18}{21}\\ &= \frac{101}{21}\\ &= 4\frac{17}{21}.\end{aligned}$$

Exercise 1.3:8
Simplifying gives

$$\begin{aligned}&5\frac{1}{2}+3\frac{2}{5}-6\frac{7}{12}\\ =\ &\frac{11}{2}+\frac{17}{5}-\frac{79}{12}\\ =\ &\frac{11\times 30+17\times 12-79\times 5}{60}\\ =\ &\frac{330+204-395}{60}\\ =\ &\frac{139}{60}\\ =\ &2\frac{19}{60}.\end{aligned}$$

(First multiply the two larger denominators by each other and check to see if the value is divisible by the other denominators).

Exercise 1.3:9
Simplifying gives

$$\begin{aligned}
& \frac{3}{4} \div \left(1\frac{5}{6} - \frac{1}{2}\right) + \frac{3}{5} \\
= & \frac{3}{4} \div \left(\frac{11}{6} - \frac{1}{2}\right) + \frac{3}{5} \\
= & \frac{3}{4} \div \left(\frac{11-3}{6}\right) + \frac{3}{5} \\
= & \frac{3}{4} \div \frac{8}{6} + \frac{3}{5} \\
= & \frac{3}{4} \times \frac{6}{8} + \frac{3}{5} \\
= & \frac{18}{32} + \frac{3}{5} \\
= & \frac{9}{16} + \frac{3}{5} \\
= & \frac{9 \times 5 + 3 \times 16}{80} \\
= & \frac{45 + 48}{80} \\
= & \frac{93}{80} \\
= & 1\frac{13}{80}.
\end{aligned}$$

Exercise 1.4:1(a)
Simplifying gives

$$2^3 = 8.$$

Exercise 1.4:1(b)
Simplifying gives

$$4^2 = 16.$$

Exercise 1.4:1(c)
Simplifying gives

$$3^{-1} = \frac{1}{3}.$$

Exercise 1.4:1(d)
Simplifying gives

$$\begin{aligned}
5^{-3} &= \frac{1}{5^3} \\
&= \frac{1}{125}.
\end{aligned}$$

Exercise 1.4:1(e)
Simplifying gives

$$\begin{aligned}
3^2 \times 3^0 &= 3^2 \times 1 \\
&= 9.
\end{aligned}$$

Exercise 1.4:1(f)
Simplifying gives

$$\begin{aligned} 2^3 \times 2^2 &= 2^{3+2} \\ &= 2^5 \\ &= 32. \end{aligned}$$

Exercise 1.4:1(g)
Simplifying gives

$$\begin{aligned} 3^2 \times 4^2 &= 9 \times 16 \\ &= 144. \end{aligned}$$

Exercise 1.4:1(h)
Simplifying gives

$$\begin{aligned} 5^{-1} \times 5^2 &= 5^{-1+2} \\ &= 5. \end{aligned}$$

Exercise 1.4:2(a)
Calculating gives

$$\begin{aligned} \sqrt{196} + \sqrt{144} &= \sqrt{14 \times 14} + \sqrt{12 \times 12} \\ &= 14 + 12 \\ &= 26. \end{aligned}$$

Exercise 1.4:2(b)
Calculating gives

$$\begin{aligned} \left(\sqrt{5}\right)^2 &= \sqrt{5} \times \sqrt{5} \\ &= \sqrt{5 \times 5} \\ &= 5. \end{aligned}$$

Exercise 1.4:2(c)
Calculating gives

$$\begin{aligned} \left(\sqrt{64}\right)^2 &= \left(\sqrt{8 \times 8}\right)^2 \\ &= (8)^2 \\ &= 8 \times 8 \\ &= 64; \end{aligned}$$

Exercise 1.4:2(d)
Calculating gives

$$\begin{aligned} \left(\sqrt{1}\right)^2 &= 1^2 \\ &= 1. \end{aligned}$$

Exercise 1.4:3(a)
Simplifying gives

$$\begin{aligned}\sqrt{100 \div 4} &= \sqrt{25} \\ &= \sqrt{5 \times 5} \\ &= 5.\end{aligned}$$

Exercise 1.4:3(b)
Simplifying gives

$$\begin{aligned}\sqrt{\frac{36}{4}} &= \sqrt{\frac{6 \times 6}{2 \times 2}} \\ &= \frac{6}{2} \\ &= 3\end{aligned}$$

or

$$\begin{aligned}\sqrt{\frac{36}{4}} &= \sqrt{9} \\ &= \sqrt{3 \times 3} \\ &= 3.\end{aligned}$$

Exercise 1.4:3(c)
Simplifying gives

$$\begin{aligned}2 \times \sqrt{144} &= 2 \times \sqrt{12 \times 12} \\ &= 2 \times 12 \\ &= 24.\end{aligned}$$

Exercise 1.4:3(d)
Simplifying gives

$$\begin{aligned}25 - \sqrt{4} &= 25 - \sqrt{2 \times 2} \\ &= 25 - 2 \\ &= 23.\end{aligned}$$

Exercise 1.4:4(a)
Calculating gives

$$\begin{aligned}\sqrt{18} &= \sqrt{2 \times 9} \\ &= \sqrt{2 \times 3 \times 3} \\ &= 3\sqrt{2}.\end{aligned}$$

Exercise 1.4:4(b)
Calculating gives

$$\begin{aligned}\frac{\sqrt{45}}{3^{-2}} &= \sqrt{3\times 3\times 5}\times 3^2\\ &= 3\sqrt{5}\times 3^2\\ &= 3\times 5^{\frac{1}{2}}\times 3^2\\ &= 3^{1+2}\times 5^{\frac{1}{2}}\\ &= 3^3\times 5^{\frac{1}{2}}\\ &= 27\sqrt{5}.\end{aligned}$$

Exercise 1.4:4(c)
Calculating gives

$$\sqrt{\frac{9-x^2}{x^2}} = \frac{(9-x^2)^{\frac{1}{2}}}{x}.$$

Exercise 1.5:1(a)
The correct symbol is

$$-5 < -2.$$

Exercise 1.5:1(b)
The correct symbol is

$$9 > -2.$$

Exercise 1.5:1(c)
The correct symbol is

$$-100 < 7.$$

Exercise 1.5:1(d)
The correct symbol is

$$-6 > -12.$$

Exercise 1.5:1(e)
The correct symbol is

$$2 > 0.$$

Exercise 1.5:1(f)
The correct symbol is

$$+3 = 3.$$

Exercise 1.5:2(a)
The correct answer is

$$0 \le x < 5.$$

Exercise 1.5:2(b)
The correct answer is

$$-3 \le x < 3.$$

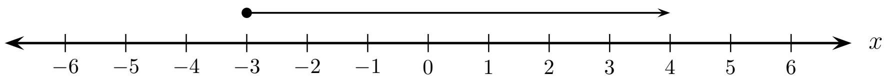
Exercise 1.5:2(c)
The correct answer is

$$x \le 5 \text{ and } x > -6.$$

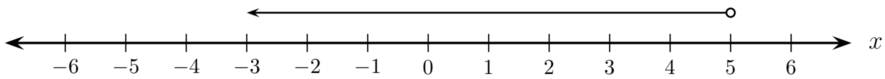
Exercise 1.5:2(d)
The correct answer is

$$x < 6 \text{ and } x \geq 0.$$

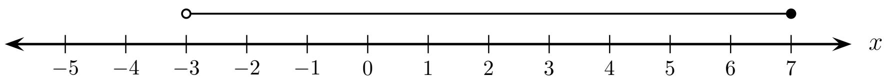
Exercise 1.5:3(a)
The correct graphical representation is



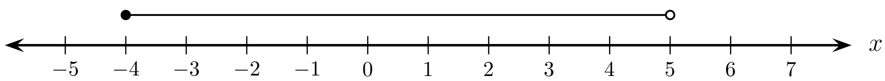
Exercise 1.5:3(b)
The correct graphical representation is



Exercise 1.5:3(c)
The correct graphical representation is



Exercise 1.5:3(d)
The correct graphical representation is



Exercise 1.5:4(a)
The solution is

$$\begin{aligned} \frac{7!}{5!} &= \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1} \\ &= 7 \times 6 \\ &= 42 \end{aligned}$$

or

$$\begin{aligned} \frac{7!}{5!} &= \frac{7 \times 6 \times 5!}{5!} \\ &= 7 \times 6 \\ &= 42. \end{aligned}$$

Exercise 1.5:4(b)
The solution is

$$\begin{aligned}(14-11)!+2!\times 4! &= 3!+2!\times 4!\\ &= 3\times 2\times 1+2\times 1\times 4\times 3\times 2\times 1\\ &= 6+2\times 24\\ &= 6+48\\ &= 54.\end{aligned}$$

Exercise 1.5:5(a)
If $x_1=3$, $x_2=5$, $x_3=4$ and $x_4=2$, then the solution is

$$\begin{aligned}\sum_{i=1}^{4} x_i &= x_1+x_2+x_3+x_4\\ &= 3+5+4+2\\ &= 14.\end{aligned}$$

Exercise 1.5:5(b)
If $x_1=3$, $x_2=5$, $x_3=4$ and $x_4=2$, then the solution is

$$\begin{aligned}\sum_{i=2}^{3} x_i &= x_2+x_3\\ &= 5+4\\ &= 9.\end{aligned}$$

Exercise 1.5:5(c)
If $x_1=3$, $x_2=5$, $x_3=4$ and $x_4=2$, then the solution is

$$\begin{aligned}\sum_{i=1}^{4} x_i^2 &= x_1^2+x_2^2+x_3^2+x_4^2\\ &= 3^2+5^2+4^2+2^2\\ &= 9+25+16+4\\ &= 54.\end{aligned}$$

Exercise 1.5:6(a)
There can be

$$26\times 26\times 26\times 26 = 456\,976$$

words.

Remember! A letter may appear more than once.

Exercise 1.5:6(b)
The number of different possible meals is

$$4\times 10\times 6 = 240.$$

Exercise 1.5:6(c)
Because the order of placement is not important, the answer is a combination:

$$\begin{aligned} {}_{12}C_3 &= \frac{12!}{9!\,3!} \\ &= \frac{12 \times 11 \times 10 \times 9!}{9! \times 3!} \\ &= \frac{12 \times 11 \times 10}{3!} \\ &= 220. \end{aligned}$$

Exercise 1.5:6(d)
In this case the order of placement is important. For example: ABCD, BCAD and DBCA all form different words. If the question had stated **groups** instead of **words** it would have been a combination. Then ABCD, BCAD and DBCA would have been the same, because the order of placement would not have been important. The answer is

$$\begin{aligned} {}_{26}P_4 &= \frac{26!}{(26-4)!} \\ &= \frac{26!}{22!} \\ &= \frac{26 \times 25 \times 24 \times 23 \times 22!}{22!} \\ &= 26 \times 25 \times 24 \times 23 \\ &= 358\,800. \end{aligned}$$

Exercise 1.6:1(a)
The area of the rectangle is calculated as

$$\begin{aligned} area &= length \times width \\ &= 25 \times 24 && \boxed{\text{mm} \times \text{mm}} \\ &= 600. && \boxed{\text{mm}^2} \end{aligned}$$

The area is $600\,\text{mm}^2$.

Exercise 1.6:1(b)
The area of the rectangle is calculated as

$$\begin{aligned} area &= length \times width \\ &= 1\,200 \times 375 && \boxed{\text{m} \times \text{m}} \\ &= 450\,000. && \boxed{\text{m}^2} \end{aligned}$$

The area is $450\,000\,\text{m}^2$.
Or

$$\begin{aligned} area &= 1{,}2 \times 0{,}375 && \boxed{\text{km} \times \text{km}} \\ &= 0{,}45. && \boxed{\text{km}^2} \end{aligned}$$

The area is $0{,}45\,\text{km}^2$.

Exercise 1.6:1(c)
The area of the rectangle is calculated as

$$\begin{aligned} area &= length \times width \\ &= 4{,}4 \times 0{,}45 && \boxed{\text{m} \times \text{m}} \\ &= 1{,}98. && \boxed{\text{m}^2} \end{aligned}$$

The area is 1,98 m^2.
Or

$$\begin{aligned} area &= 4\,400 \times 450 && \boxed{\text{mm} \times \text{mm}} \\ &= 1\,980\,000. && \boxed{\text{mm}^2} \end{aligned}$$

The area is 1 980 000 mm^2.

Exercise 1.6:1(d)
The area of the rectangle is calculated as

$$\begin{aligned} area &= length \times width \\ &= 225 \times 122 && \boxed{\text{mm} \times \text{mm}} \\ &= 27\,450. && \boxed{\text{mm}^2} \end{aligned}$$

The area is 27 450 mm^2.

Exercise 1.6:2(a)
The area of 1 km^2 converted to m^2 is

$$\begin{aligned} area &= length \times width \\ &= 1 \times 1 && \boxed{\text{km} \times \text{km}} \\ &= 1\,000 \times 1\,000 && \boxed{\text{Convert km to m to get m} \times \text{m.}} \\ &= 1\,000\,000. && \boxed{\text{m}^2} \end{aligned}$$

Therefore, 1 km^2 equals 1 000 000 m^2.

Exercise 1.6:2(b)
The area of 1 m^2 converted to mm^2 is

$$\begin{aligned} area &= length \times width \\ &= 1 \times 1 && \boxed{\text{m} \times \text{m}} \\ &= 1\,000 \times 1\,000 && \boxed{\text{Convert m to mm to get mm} \times \text{mm.}} \\ &= 1\,000\,000. && \boxed{\text{mm}^2} \end{aligned}$$

Therefore, 1 m^2 equals 1 000 000 mm^2.

Exercise 1.6:2(c)
The area of $1\,\text{cm}^2$ converted to mm^2 is

$$\begin{aligned} area &= length \times width \\ &= 1 \times 1 && \boxed{\text{cm} \times \text{cm}} \\ &= 10 \times 10 && \boxed{\text{Convert cm to mm to get mm} \times \text{mm.}} \\ &= 100. && \boxed{\text{mm}^2} \end{aligned}$$

Therefore, $1\,\text{cm}^2$ equals $100\,\text{mm}^2$.

Exercise 1.6:2(d)
The area of 1 ha converted to m^2 is

$$\begin{aligned} area &= length \times width \\ &= 100 \times 100 && \boxed{\text{m} \times \text{m}} \\ &= 10\,000. && \boxed{\text{m}^2} \end{aligned}$$

Therefore, 1 ha equals $10\,000\,\text{m}^2$.

Exercise 1.6:2(e)
The area of $1\,\text{m}^2$ converted to cm^2 is

$$\begin{aligned} area &= length \times width \\ &= 1 \times 1 && \boxed{\text{m} \times \text{m}} \\ &= 100 \times 100 && \boxed{\text{Convert m to cm to get cm} \times \text{cm.}} \\ &= 10\,000. && \boxed{\text{cm}^2} \end{aligned}$$

Therefore, $1\,\text{m}^2$ equals $10\,000\,\text{cm}^2$.

Exercise 1.6:2(f)
The area of $1\,\text{cm}^2$ converted to m^2 is

$$\begin{aligned} area &= length \times width \\ &= 1 \times 1 && \boxed{\text{cm} \times \text{cm}} \\ &= 0{,}01 \times 0{,}01 && \boxed{\text{Convert cm to m to get m} \times \text{m.}} \\ &= 0{,}0001. && \boxed{\text{m}^2} \end{aligned}$$

Therefore, $1\,\text{cm}^2$ equals $0{,}0001\,\text{m}^2$. Consider the ratio

$$\begin{array}{rcl} \text{cm}^2 & : & \text{m}^2 \\ 1 & : & 0{,}0001 \\ 1 \times 24{,}6 & = & 0{,}0001 \times 24{,}6 \\ 24{,}6 & = & 0{,}00246. \end{array}$$

Therefore, $24{,}6\,\text{cm}^2$ equals $0{,}00246\,\text{m}^2$.

Exercise 1.6:2(g)
The area of $1\,\text{mm}^2$ converted to m^2 is

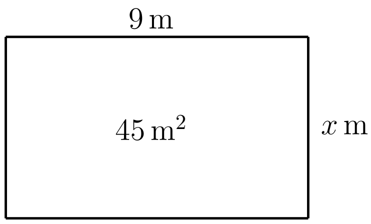
$$\begin{aligned} area &= length \times width \\ &= 1 \times 1 && \boxed{\text{mm} \times \text{mm}} \\ &= 0{,}001 \times 0{,}001 && \boxed{\text{Convert mm to m to get m} \times \text{m.}} \\ &= 0{,}000001. && \boxed{\text{m}^2} \end{aligned}$$

Therefore, $1\,\text{mm}^2$ equals $0{,}000001\,\text{m}^2$. Consider the ratio

$$\begin{aligned} \text{mm}^2 &\;:\; \text{m}^2 \\ 1 &\;:\; 0{,}000001 \\ 1 \times 24\,869{,}3 &= 0{,}000001 \times 24\,869{,}3 \\ 24\,869{,}3 &= 0{,}0248693. \end{aligned}$$

Therefore, $24\,869{,}3\,\text{mm}^2$ equals $0{,}0248693\,\text{m}^2$ or $2{,}48693 \times 10^{-2}\,\text{m}^2$.

Exercise 1.6:3
A sketch of the rectangle (where the width is x m) is given below.



Use the area formula to determine the value of x :

$$\begin{aligned} length \times width &= area \\ 9 \times x &= 45 && \boxed{\text{m} \times \text{m} = \text{m}^2} \\ 9x &= 45 \\ \frac{9x}{9} &= \frac{45}{9} && \boxed{\frac{\text{m}^2}{\text{m}}} \\ x &= 5. && \boxed{\text{m}} \end{aligned}$$

The width is 5 m.

Exercise 1.6:4
The volume of the container is calculated as

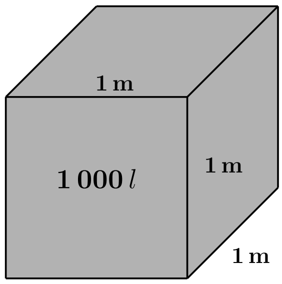
$$\begin{aligned} volume &= length \times width \times height \\ &= 1 \times 1 \times 1 && \boxed{\text{m} \times \text{m} \times \text{m}} \\ &= 1. && \boxed{\text{m}^3} \end{aligned}$$

The volume is $1\,\text{m}^3$.
As seen in paragraph 1.7.3, to convert from m^3 to litres we must multiply by 1 000 or 10^3:

$$1 \times 1\,000 = 1\,000. \qquad \boxed{\text{m}^3 \times 1\,000 = l}$$

Thus $1\,000\,l$ can be poured into the container.

This can also be illustrated as follows:



The volume of the cube (converted from m^3 to cm^3) is calculated as

$$\begin{aligned} volume &= length \times width \times height \\ &= 1 \times 1 \times 1 && \boxed{\text{m} \times \text{m} \times \text{m}} \\ &= 100 \times 100 \times 100 && \boxed{\text{Convert m to cm to get cm} \times \text{cm} \times \text{cm}} \\ &= 1\,000\,000. && \boxed{\text{cm}^3} \end{aligned}$$

The volume of the cube is $1\,000\,000\,\text{cm}^3$ or $1\,000\,000\,\text{m}l$. (Remember $1\,\text{cm}^3 = 1\,\text{m}l$.)
There are $1\,000\,\text{m}l$ in one litre.
Thus, the volume of the cube (in litres) is calculated as

$$\frac{1\,000\,000}{1\,000} = 1\,000. \qquad \boxed{\frac{\text{m}l}{1\,000} = l}$$

Therefore, the volume of the cube is $1\,000\,l$.

Exercise 1.6:5
Determine the volume of the fuel tank:

$$\begin{aligned} volume &= length \times width \times height \\ &= 60 \times 50 \times 20 && \boxed{\text{cm} \times \text{cm} \times \text{cm}} \\ &= 60\,000. && \boxed{\text{cm}^3} \end{aligned}$$

The volume of the fuel tank is $60\,000\,\text{cm}^3$.

Remember: $1\,l = 10^3\,\text{cm}^3$ or $1\,000\,\text{cm}^3$. Therefore, the volume of the fuel tank is $60\,l$.

The fuel costs R9,16 per litre. The cost to fill the tank is calculated as

$$9{,}16 \times 60 = 549{,}60. \qquad \boxed{\text{rand/litre} \times \text{litres} = \text{rand}}$$

It will cost R549,60 to fill the tank.

Component 2. Collection, presentation and description of data

Exercise 2.1:1
The smallest value is 28 and the largest is 48. The range is

$$R = 48 - 28 = 20.$$

The measure for the number of intervals is

$$\frac{R}{10} = \frac{20}{10} = 2,$$

but, in my opinion, a frequency table with only two intervals is not very useful. If I use five intervals the width of each interval is

$$\frac{20}{5} = 4.$$

The lower bound of the first interval is half a unit less than the smallest value of 28. Then the first interval is 27,5 – 31,5. A quick calculation

$$(5(\text{intervals}) \times 4(\text{width}) = 20 \text{ and } 20 + 27{,}5 = 47{,}5)$$

gives an upper limit for the last interval of 47,5 which poses a problem. So I will use six intervals:

$$\frac{20}{6} = 3\frac{1}{3},$$

hence I choose the width as four. Now the first interval is 27,5 – 31,5. My quick calculation now gives

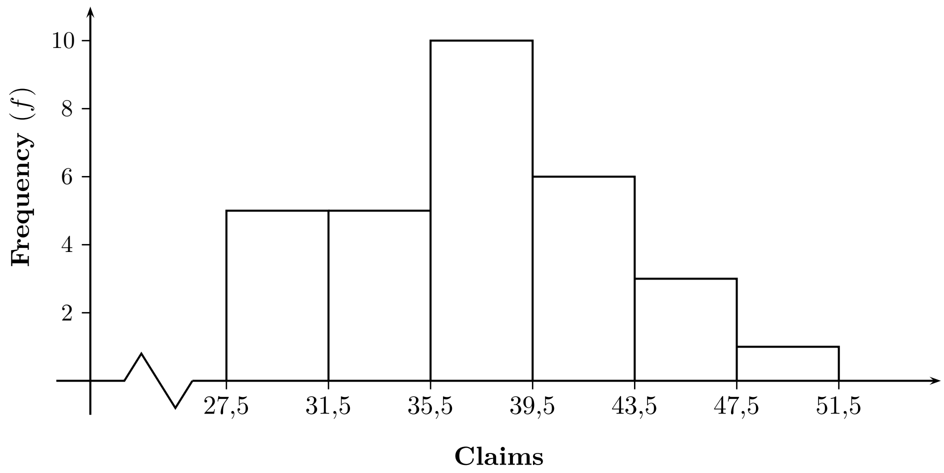
$$6 \times 4 + 27{,}5 = 51{,}5$$

for the last interval's upper bound!

The frequency table of the data is given below.

Interval		Frequency
27,5 – 31,5	~~||||~~	5
31,5 – 35,5	~~||||~~	5
35,5 – 39,5	~~||||~~ ~~||||~~	10
39,5 – 43,5	~~||||~~ |	6
43,5 – 47,5	|||	3
47,5 – 51,5	|	1
		30

The histogram of the insurance claims data is given below.



Exercise 2.1:2
The cumulative frequency table is as follows:

Upper limit	Cumulative frequency
$<$ 31,5	5
$<$ 35,5	10
$<$ 39,5	20
$<$ 43,5	26
$<$ 47,5	29
$<$ 51,5	30

(With the cumulative frequency table we may use either "smaller than" or "smaller than or equal to" ($<$ or $\leq$).)

Exercise 2.1:3
The stem-and-leaf diagram is

Stem	Leaf	Frequency
2	8	1
3	0 0 1 1 3 4 4 4 5 6 6 7 7 8 8 8 9 9 9	19
4	0 0 0 1 1 2 4 5 6 8	10

The values (or leaves) within a stem may be separated so that the values from 0 to 4 are separated from the values from 5 to 9.

This gives more information and looks like this:

Stem	Leaf	Frequency
2	8	1
3	0 0 1 1 3 4 4 4	8
3	5 6 6 7 7 8 8 8 9 9 9	11
4	0 0 0 1 1 2 4	7
4	5 6 8	3

Exercise 2.1:4(a)
The minimum number of claims processed per week is 28. If a worker processes only 26 claims per week for a whole month the chances are good that there is a problem.

Exercise 2.1:4(b)
In the interval 35,5 – 39,5 the frequency is 10. Therefore,

$$\frac{10}{30} \times 100 = 33\frac{1}{3}\%$$

of the workers can process 36 to 39 claims per week. Is the competitor more productive? This depends on whether the claims being processed are similar. More information should be obtained before I take action.

Exercise 2.1:4(c)
Considering the frequency table, the number of claims of 36 or less is accounted for by the first two intervals. The sum of the frequencies is

$$5 + 10 = 15.$$

Expressed as a percentage it gives

$$\frac{15}{30} \times 100 = 50\%.$$

Therefore, 50% of the workers processes less than 36 claims per week. I would transfer some of the personnel to other divisions.

Exercise 2.2:1
The mean is

$$\begin{aligned} \overline{x} &= \frac{\sum_{i=1}^{10} x_i}{n} \\ &= \frac{1\,671}{10} \\ &= 167{,}1. \end{aligned}$$

Exercise 2.2:2
The data must be in ascending order before the median can be determined:

104; 127; 131; 135; 146; 170; 175; 179; 190; 314

The median is the observation in the

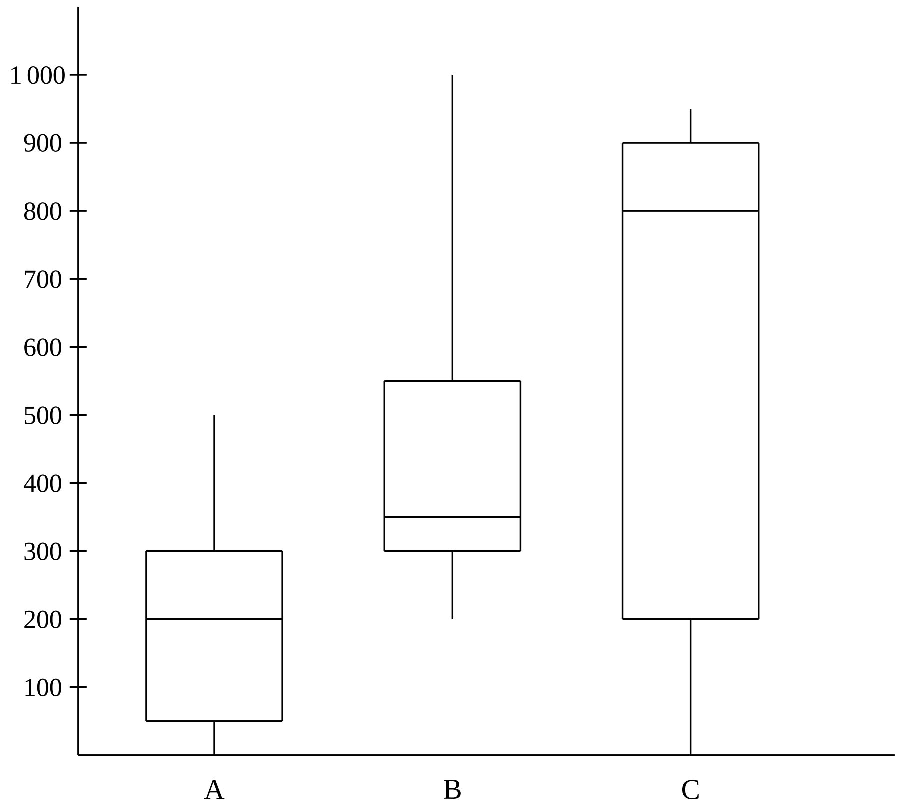
$$\begin{aligned} \frac{n+1}{2} &= \frac{10+1}{2} \\ &= 5{,}5\text{th} \end{aligned}$$

position, that is the 5,5th observation. Because there is no such value, we calculate the mean of the fifth and sixth values. The median is

$$
\begin{aligned}
Me &= \frac{146 + 170}{2} \\
&= 158.
\end{aligned}
$$

Exercise 2.2:3
The mode is the value that occurs the most often. There is no value that occurs more than once and there is no mode.

Exercise 2.3
The box-and-whiskers diagram is as follows:



It is clear that region C has the highest median, but also the largest interquartile deviation. The inner 50% of the rainfall figures is spread rather widely.

The spreads for A and B are quite similar. However, whereas the data for A are more or less symmetrical to the median, the distribution for B is rather skew. In B 25% of the figures fall between 300 and 350, and 25% of the figures fall between 350 and 550. The figures are more "concentrated" between 300 and 350. The distribution for C is also very skew, with 25% of the figures falling between 800 and 900 and 25% of the figures falling between 200 and 800.

Component 3. Index numbers and transformations

Exercise 3.1:1
The table is given below.

p_{2009}	q_{2009}	p_{2012}	q_{2012}	$p_{2009} \times q_{2009}$	$p_{2009} \times q_{2012}$	$p_{2012} \times q_{2009}$	$p_{2012} \times q_{2012}$
34,50	200	41,00	200	6 900,0	6 900	8 200	8 200
10,20	1 000	11,75	2 200	10 200,0	22 440	11 750	25 850
9,50	365	10,00	550	3 467,5	5 225	3 650	5 500
				20 567,5	34 565	23 600	39 550

The Laspeyres price index for 2012 with 2009 as base year is

$$\begin{aligned} P_L(n) &= \frac{\sum p_n q_0}{\sum p_0 q_0} \times 100 \\ P_L(2012) &= \frac{\sum p_{2012} q_{2009}}{\sum p_{2009} q_{2009}} \times 100 \\ &= \frac{23\,600}{20\,567{,}5} \times 100 \\ &= 114{,}74. \end{aligned}$$

The Paasche price index for 2012 with 2009 as base year is

$$\begin{aligned} P_P(n) &= \frac{\sum p_n q_n}{\sum p_0 q_n} \times 100 \\ P_P(2012) &= \frac{\sum p_{2012} q_{2012}}{\sum p_{2009} q_{2012}} \times 100 \\ &= \frac{39\,550}{34\,565} \times 100 \\ &= 114{,}42. \end{aligned}$$

Exercise 3.1:2
The table is given below.

p_{2010}	q_{2010}	p_{2012}	q_{2012}	$p_{2010} \times q_{2010}$	$p_{2010} \times q_{2012}$	$p_{2012} \times q_{2010}$	$p_{2012} \times q_{2012}$
39,20	150	41,00	200	5 880	7 840,0	6 150	8 200
10,50	2 000	11,75	2 200	21 000	23 100,0	23 500	25 850
9,75	400	10,00	550	3 900	5 362,5	4 000	5 500
				30 780	36 302,5	33 650	39 550

The Laspeyres quantity index for 2012 with 2010 as base year is

$$\begin{aligned} Q_L(n) &= \frac{\sum p_0 q_n}{\sum p_0 q_0} \times 100 \\ Q_L(2012) &= \frac{\sum p_{2010} q_{2012}}{\sum p_{2010} q_{2010}} \times 100 \\ &= \frac{36\,302{,}5}{30\,780} \times 100 \\ &= 117{,}94. \end{aligned}$$

The Paasche quantity index for 2012 with 2010 as base year is

$$\begin{aligned} Q_P(n) &= \frac{\sum p_n q_n}{\sum p_n q_0} \times 100 \\ Q_P(2012) &= \frac{\sum p_{2012} q_{2012}}{\sum p_{2012} q_{2010}} \times 100 \\ &= \frac{39\,550}{33\,650} \times 100 \\ &= 117{,}53. \end{aligned}$$

Exercise 3.1:3
The value index for 2012 with 2009 as base year is

$$\begin{aligned} V &= \frac{\sum p_n q_n}{\sum p_0 q_0} \times 100 \\ &= \frac{\sum p_{2012} q_{2012}}{\sum p_{2009} q_{2009}} \times 100 \\ &= \frac{39\,550}{20\,567{,}5} \times 100 \\ &= 192{,}29. \end{aligned}$$

Exercise 3.2:1
Consider the following table:

Year	Wage (R)	CPI (base year is 2010)	Deflated wage at value of R in 2010
2008	5 050	97,08	$\frac{5\,050}{97{,}08} \times 100 = 5\,201{,}90$
2010	6 020	100,00	$\frac{6\,020}{100{,}00} \times 100 = 6\,020{,}00$

The following statement is made in terms of the value of the rand in 2010. Although he receives R970 (ie 6 020 − 5 050) per month more in 2010, the purchasing power of his wages has increased by only R818,10 (ie 6 020 − 5 201,90).

Exercise 3.2:2
An amount of R8,12 buys $1,00.

An amount of R4 000,00 buys

$$\frac{4\,000}{8{,}12} = 492{,}61$$

dollar, that is $492,61.

Exercise 3.2:3
We know from the activity in paragraph 5.2.3 that there are 32,1507 fine ounces in one kilogram of gold. The gold price for one kilogram of gold in dollar is

$$933{,}80 \times 32{,}1507 = 30\,022{,}32.$$

That is $30 022,32 per kilogram.

If $1,00 = R8,12 then the value of $30 022,32 in rand is

$$30\,022{,}32 \times 8{,}12 = 243\,781{,}27.$$

Therefore, the rand value of one kilogram of gold is R243 781,27.

Exercise 3.2:4
The period is three years.

The growth rate from 2008 to 2011 is

$$\begin{aligned}\left[\left(\frac{BBP_n}{BBP_0}\right)^{\frac{1}{3}} - 1\right] \times 100 &= \left[\left(\frac{BBP_{2011}}{BBP_{2008}}\right)^{\frac{1}{3}} - 1\right] \times 100 \\ &= \left[\left(\frac{558\,760}{517\,178}\right)^{\frac{1}{3}} - 1\right] \times 100 \\ &= 2{,}61\% \text{ per year.}\end{aligned}$$

Component 4. Functions and representations of functions

Exercise 4.1:1(a)
The equation of the straight line is $y = ax + b$.
Determine a:
$$a = \frac{y_2 - y_1}{x_2 - x_1}.$$

Two points $(1; 2)$ and $(3; 3)$ are given. Select any one of the two points to be $(x_1;\, y_1)$ and the other one to be $(x_2;\, y_2)$. Let $(1;\, 2) = (x_1;\, y_1)$ and $(3;\, 3) = (x_2;\, y_2)$.

Then
$$\begin{aligned} a &= \frac{3-2}{3-1} \\ &= \frac{1}{2} \\ &= 0{,}5. \end{aligned}$$

Thus
$$y = 0{,}5x + b.$$

Determine b.

Take any one of the two points and substitute the values for x and y into the equation $y = 0{,}5x + b$. Say we choose the point $(1; 2)$:
$$\begin{aligned} y &= 0{,}5x + b \\ 2 &= 0{,}5 \times 1 + b \\ 2 &= 0{,}5 + b \\ 2 - 0{,}5 &= b \\ b &= 1{,}5. \end{aligned}$$

Thus the equation of the line that cuts through the points $(1; 2)$ and $(3; 3)$ is
$$y = 0{,}5x + 1{,}5.$$

Exercise 4.1:1(b)
Determine the intercepts on the x- and y-axes for $y = 0{,}5x + 1{,}5$.

The intercept on the x-axis is where the line cuts through the x-axis; meaning the x-value where the y-value is 0, is
$$\begin{aligned} 0 &= 0{,}5x + 1{,}5 \\ 0 - 1{,}5 &= 0{,}5x \\ 0{,}5x &= -1{,}5 \\ x &= \frac{-1{,}5}{0{,}5} \\ &= -3. \end{aligned}$$

The point is $(-3; 0)$.

The intercept on the y-axis is where the line cuts through the y-axis; where $x = 0$ is
$$\begin{aligned} y &= 0{,}5 \times 0 + 1{,}5 \\ y &= 1{,}5. \end{aligned}$$

The point is $(0; 1{,}5)$.

Exercise 4.1:1(c)
Two lines are parallel if the slopes of the two lines are the same.

The line in (a) is

$$y = 0{,}5x + 1{,}5$$

with slope 0,5.

The line in (c) is

$$y = 2 + x$$

with slope 1.

The two lines are not parallel.

Exercise 4.1:1(d)
For the line in (a): Plot the two points $(1; 2)$ and $(3; 3)$ and draw a line through them.

For the line in (c): Calculate two points on the line $y = 2 + x$.

If $x = 0$ then

$$\begin{aligned} y &= 2 + 0 \\ &= 2. \end{aligned}$$

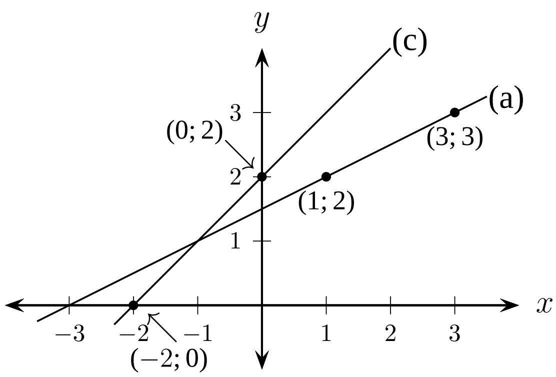
The point is (0; 2).

If $y = 0$ then

$$\begin{aligned} 0 &= 2 + x \\ x &= -2. \end{aligned}$$

The point is (−2; 0).

Plot the two points and draw a line through the points. The graph is given below.



Exercise 4.1:2
For the line $y = 5 + 2x$ the intercepts are

on the y-axis (where $x = 0$):

$$y = 5,$$

on the x-axis (where $y = 0$):

$$x = \frac{-5}{2}.$$

The points are $(0; 5)$ and $\left(-2\frac{1}{2}; 0\right)$.

For the line $y = 2 + x$ the intercepts are $(0; 2)$ and $(-2; 0)$, as seen in 2.1:1(d).

The lines are not parallel since the slopes are different, namely 2 and 1.

At $x = 3{,}5$ the y-values for the two lines are respectively

$$\begin{aligned} y &= 5 + 2 \times 3{,}5 \\ &= 12 \end{aligned}$$

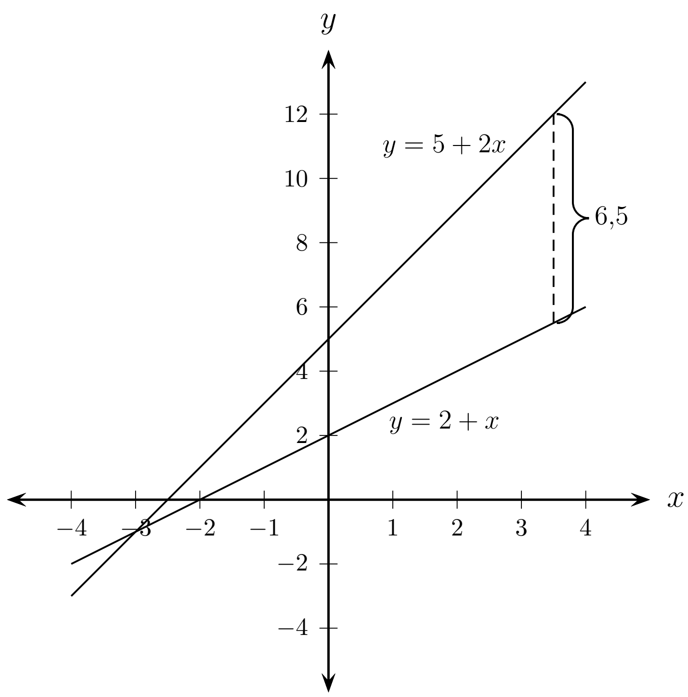
and

$$\begin{aligned} y &= 2 + 3{,}5 \\ &= 5{,}5. \end{aligned}$$

The vertical distance between the two lines at $x = 3{,}5$ is

$$12 - 5{,}5 = 6{,}5.$$

This is illustrated on the graph below.



Exercise 4.1:3
The line $x = 2$ is a straight line through 2 on the x-axis parallel to the y-axis.

For the line $y = 4x$:
if $x = 0$, then

$$\begin{aligned} y &= 4 \times 0 \\ &= 0, \end{aligned}$$

if $x = 1$, then

$$\begin{aligned} y &= 4 \times 1 \\ &= 4. \end{aligned}$$

Two points on the line are $(0;0)$ and $(1;4)$.

For the line $y=-2x-3$:
if $x=0$, then

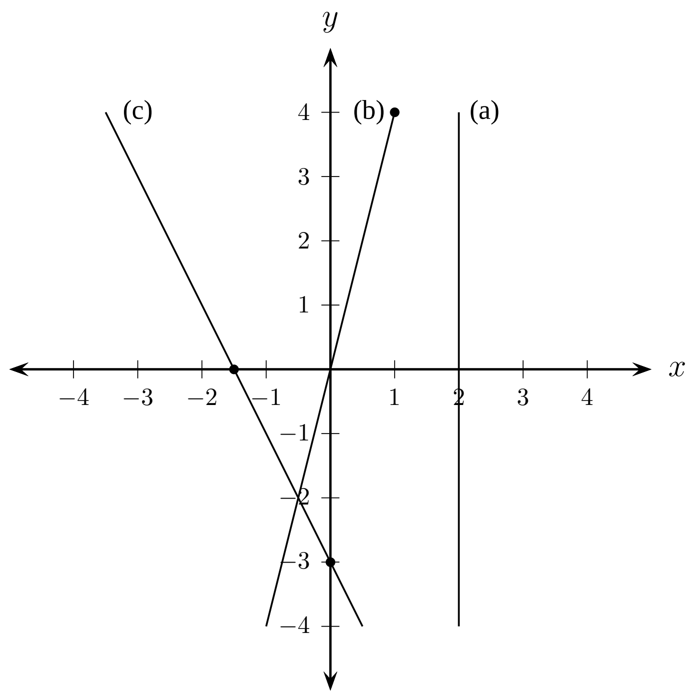
$$\begin{aligned} y &= -2\times 0-3 \\ &= -3, \end{aligned}$$

if $y=0$, then

$$\begin{aligned} 0 &= -2x-3 \\ 2x &= -3 \\ x &= -\frac{3}{2}. \end{aligned}$$

Two points on the line are $(-\frac{3}{2};\ 0)$ and $(0;-3)$.

The graph is given below.



Exercise 4.1:4
Let p represent the price and x the number of customers.

The point $(x;\ p)$, that is $(60;\ 6\,000)$, is given.

The general expression is

$$y=ax+b$$

or, written in terms of our variables,

$$p=ax+b.$$

The slope a gives you the rate of change in the p-value for a unit change in x-value. It is given that if the price increases by R500 the number of customers decreases by 3. Thus

$$a=\frac{500}{-3}.$$

The general expression reduces to

$$p = \frac{500}{-3}x + b.$$

How do we find the value of b? Since the line must pass through the given point (60; 6 000), substitute the x- and p-value into the last expression. This gives

$$\begin{aligned} 6\,000 &= \frac{500}{-3} \times 60 + b \\ 6\,000 &= -10\,000 + b \\ b &= 16\,000. \end{aligned}$$

The expression for the line is therefore

$$p = \frac{500}{-3}x + 16\,000$$

or

$$p = -166{,}67x + 16\,000.$$

Exercise 4.2:1(a)
From

$$y = -0{,}4x^2 + 0{,}2x + 1{,}2$$

we have that

$$a = -0{,}4,\ b = 0{,}2 \text{ and } c = 1{,}2.$$

Since $a < 0$, the function has a minimum.
The value of x at the vertex is

$$\begin{aligned} x_m &= \frac{-b}{2a} \\ &= \frac{-0{,}2}{2(-0{,}4)} \\ &= \frac{-0{,}2}{-0{,}8} \\ &= 0{,}25. \end{aligned}$$

The maximum value of the function is

$$\begin{aligned} y &= f\,(0{,}25) \\ &= -0{,}4\,(0{,}25)^2 + 0{,}2\,(0{,}25) + 1{,}2 \\ &= 1{,}225. \end{aligned}$$

The intercept on the y-axis is at

$$c = 1{,}2.$$

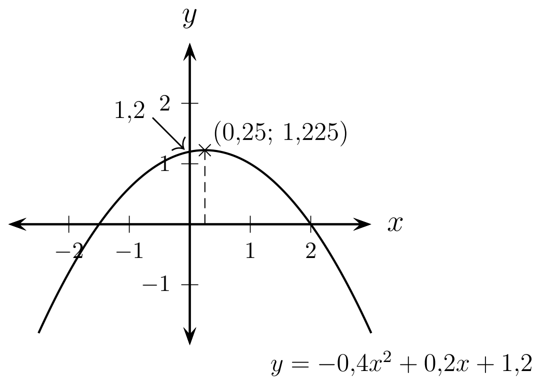
The discriminant is

$$\begin{aligned} b^2 - 4ac &= 0{,}2^2 - 4 \times (-0{,}4) \times (1{,}2) \\ &= 1{,}96. \end{aligned}$$

The intercepts on the x-axis are

$$\begin{aligned} x &= \frac{-0{,}2 - \sqrt{1{,}96}}{2(-0{,}4)} \qquad \text{and} \qquad & x &= \frac{-0{,}2 + \sqrt{1{,}96}}{2(-0{,}4)} \\ &= 2 & &= -1{,}5. \end{aligned}$$

The quadratic is sketched below.



Exercise 4.2:1(b)
From

$$y = -x^2 - 2x - 1$$

we have that

$$a = -1,\ b = -2 \text{ and } c = -1.$$

Since $a < 0$, the function has a maximum.
The value of x at the vertex is

$$\begin{aligned} x_m &= \frac{-b}{2a} \\ &= \frac{-(-2)}{2(-1)} \\ &= -1. \end{aligned}$$

The maximum value of the function is

$$\begin{aligned} y &= f(-1) \\ &= -(-1)^2 - 2(-1) - 1 \\ &= -1 + 2 - 1 \\ &= 0. \end{aligned}$$

The intercept on the y-axis is at

$$c = -1.$$

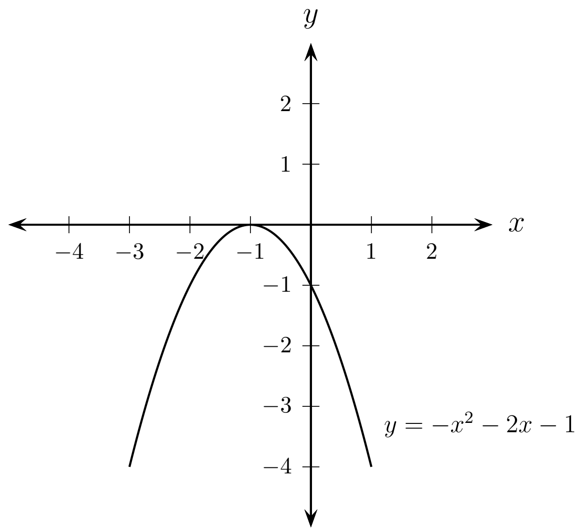
The discriminant is

$$b^2 - 4ac = 0.$$

The intercepts on the x-axis are the same as the value of x at the vertex:

$$\begin{aligned} x &= \frac{-(-2) \pm \sqrt{0}}{2(-1)} \\ &= \frac{2}{-2} \\ &= -1. \end{aligned}$$

Thus the parabola just touches the x-axis at $x_m = -1$.
The sketch appears below.



Exercise 4.2:1(c)
From

$$y = x^2 + 4x + 5$$

we have that

$$a = 1,\ b = 4 \text{ and } c = 5.$$

Since $a > 0$, the function has a minimum.
The value of x at the vertex is

$$\begin{aligned} x_m &= \frac{-b}{2a} \\ &= \frac{-4}{2(1)} \\ &= -2. \end{aligned}$$

The minimum value is

$$\begin{aligned} y &= f(-2) \\ &= (-2)^2 + 4(-2) + 5 \\ &= 1. \end{aligned}$$

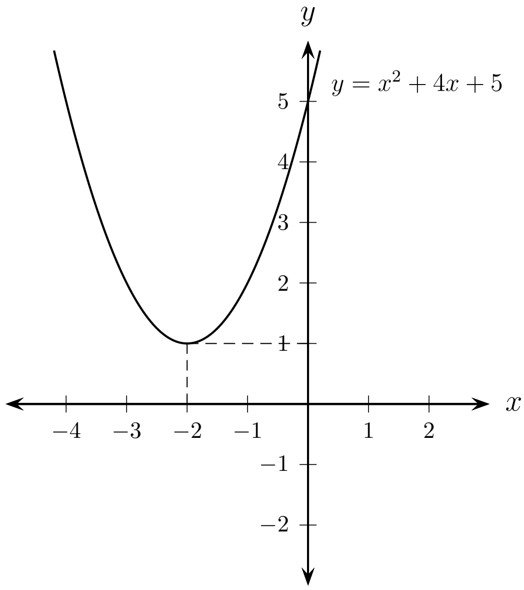
The intercept on the y-axis is

$$c = 5.$$

The discriminant is

$$\begin{aligned} b^2 - 4ac &= 4^2 - 4(1)(5) \\ &= 16 - 20 \\ &= -4 \end{aligned}$$

which is < 0. There are no intercepts on the x-axis. The sketch is given below.



Exercise 4.2:1(d)
From

$$y = -x^2 + 9$$

we have that

$$a = -1,\ b = 0 \text{ and } c = 9.$$

Since $a < 0$, the function has a maximum value.
The value of x at the vertex is

$$\begin{aligned} x_m &= \frac{-b}{2a} \\ &= \frac{-0}{2(-1)} \\ &= 0. \end{aligned}$$

The maximum value of the function is

$$\begin{aligned} y &= f(0) \\ &= -(0)^2 + 9 \\ &= 9. \end{aligned}$$

The intercept on the y-axis is at

$$c = 9.$$

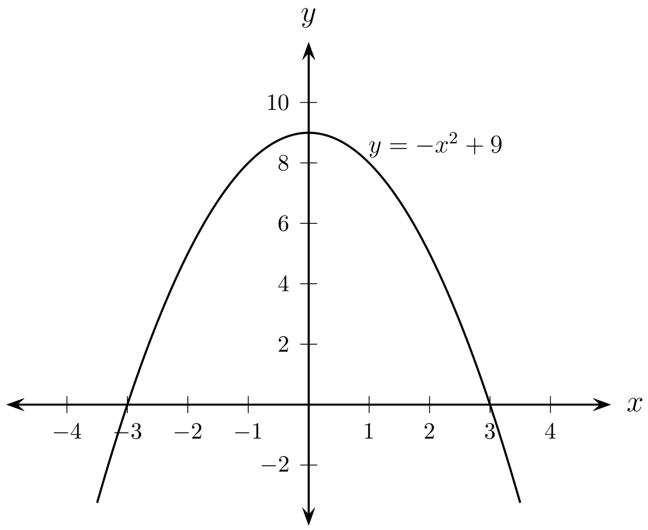
The discriminant is

$$\begin{aligned} b^2 - 4ac &= 0^2 - 4(-1)(9) \\ &= 36. \end{aligned}$$

The intercepts on the x-axis are

$$x = \frac{0-\sqrt{36}}{2(-1)} = \frac{-6}{-2} = 3 \quad \text{and} \quad x = \frac{0+\sqrt{36}}{2(-1)} = \frac{6}{-2} = -3.$$

The quadratic is sketched below.



Exercise 4.2:2
From

$$d = p^2 - 45p + 520$$

we have that

$$a = 1,\ b = -45 \text{ and } c = 520.$$

The price at the minimum is

$$p = \frac{-(-45)}{2(1)} = 22{,}5.$$

Thus, the price that minimises the weekly demand is R22,50 per litre.

The demand at the minimum is

$$\begin{aligned} d &= f(p) \\ &= f(22{,}5) \\ &= 22{,}5^2 - 45(22{,}5) + 520 \\ &= 13{,}75. \end{aligned}$$

The minimum demand will be 13,75 litres for a price of R22,50 per litre.

Component 5. Linear systems

Exercise 5.1:1
The equations are

$$7x + 5y = -4 \qquad (1)$$

and

$$3x + 4y = 2. \qquad (2)$$

From (1):

$$\begin{aligned} 7x &= -4 - 5y \\ x &= -\frac{4}{7} - \frac{5}{7}y. \qquad (3) \end{aligned}$$

Substitute (3) into (2):

$$\begin{aligned} 3\left(-\frac{4}{7} - \frac{5}{7}y\right) + 4y &= 2 \\ -\frac{12}{7} - \frac{15}{7}y + 4y &= 2 \\ -\frac{15}{7}y + \frac{28}{7}y &= 2 + \frac{12}{7} \\ \frac{13}{7}y &= \frac{26}{7} \\ 13y &= 26 \\ y &= 2. \end{aligned}$$

Substitute $y = 2$ into (3):

$$\begin{aligned} x &= -\frac{4}{7} - \frac{5}{7} \times 2 \\ &= \frac{-14}{7} \\ &= -2. \end{aligned}$$

The solution is the point

$$(-2;\, 2).$$

Exercise 5.1:2
The equations are

$$2x + 2y = 3 \qquad (1)$$

and

$$5x + \frac{y}{2} = -6. \qquad (2)$$

From (1):

$$\begin{aligned} 2x &= 3 - 2y \\ x &= \frac{3}{2} - \frac{2y}{2} \\ &= \frac{3}{2} - y. \qquad (3) \end{aligned}$$

Substitute (3) into (2):

$$\begin{aligned}
5\left(\frac{3}{2}-y\right)+\frac{y}{2} &= -6\\
\frac{15}{2}-5y+\frac{y}{2} &= -6\\
-\frac{10y}{2}+\frac{y}{2} &= -6-\frac{15}{2}\\
-\frac{9y}{2} &= -\frac{27}{2}\\
-9y &= -27\\
y &= 3.
\end{aligned}$$

Substitute $y = 3$ into (3):

$$\begin{aligned}
x &= \frac{3}{2}-3\\
&= \frac{3}{2}-\frac{6}{2}\\
&= -\frac{3}{2}.
\end{aligned}$$

The solution is

$$\left(-\frac{3}{2};\,3\right).$$

Exercise 5.1:3
The equations are

$$x + 4y = 49 \qquad (1)$$

and

$$-2x + \ y = \ 1. \qquad (2)$$

From (1):

$$x = 49 - 4y. \qquad (3)$$

Substitute (3) into (2):

$$\begin{aligned}
-2\,(49-4y)+y &= 1\\
-98+8y+y &= 1\\
9y &= 98+1\\
9y &= 99\\
y &= 11.
\end{aligned}$$

Substitute $y = 11$ into (3):

$$\begin{aligned}
x &= 49-4\times 11\\
&= 49-44\\
&= 5.
\end{aligned}$$

The solution is

$$(5;\,11)\,.$$

Exercise 5.2:1 The inequality is

$$11 \geq 6 - 4x.$$

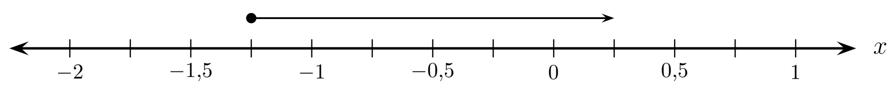
Subtract 6:

$$5 \geq -4x.$$

Divide by -4:

$$\begin{aligned} -\frac{5}{4} &\leq x \\ x &\geq -\frac{5}{4}. \end{aligned}$$

Thus, the solution is all values of x greater or equal to $-\frac{5}{4}$ (that is $-1{,}25$), as shown.



Exercise 5.2:2
The inequality is

$$4x + 4 < 1{,}5x - 6.$$

Subtract 4:

$$4x < 1{,}5x - 10.$$

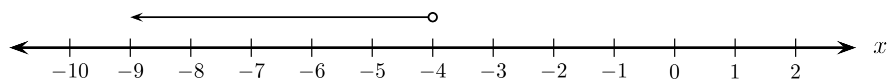
Subtract $1{,}5x$:

$$2{,}5x < -10.$$

Divide by 2,5:

$$x < -4.$$

Thus, the solution consists of all values of x less than -4, as shown.



Exercise 5.3:1
The inequality is

$$3x + y - 3 > 0.$$

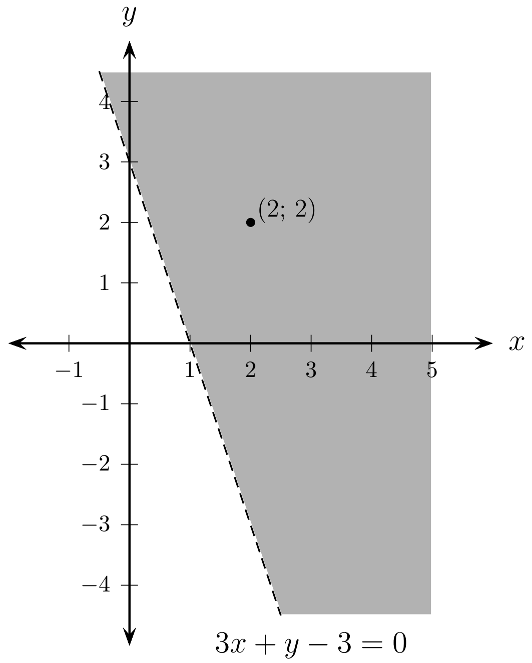
The corresponding straight line is

$$3x + y - 3 = 0$$

or

$$y = -3x + 3.$$

This is depicted below.



$3x + y - 3 = 0$

Now consider the point $(2; 2)$, which is to the right of and above the line. Substituted in the left-hand side of the inequality, $3x + y - 3 > 0$, it gives

$$3 \times 2 + 2 - 3 = 5$$

and

$$5 > 0.$$

Thus all the points to the right and above the line satisfy the inequality. Since the inequality contains no $=$ sign, the points on the line do not satisfy it. We indicate this by drawing a dashed line.

Exercise 5.3:2
The inequality is

$$2x + 4y + 1 \leq x + y - 2.$$

Add $-x - y + 2$ to both sides:

$$\begin{array}{rcl} 2x - x + 4y - y + 1 + 2 & \leq & 0 \\ x + 3y + 3 & \leq & 0. \end{array}$$

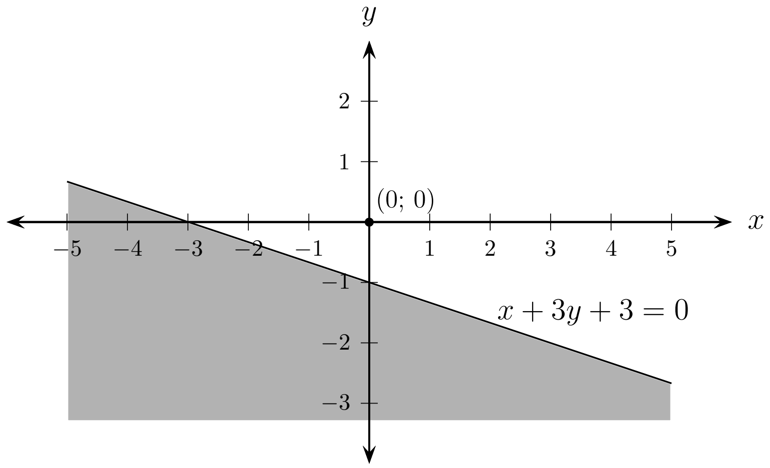
The corresponding straight line is

$$x + 3y + 3 = 0$$

or

$$y = -\frac{1}{3}x - 1.$$

This is shown in the graph.



Consider the point $(0;0)$. Substitute it in the inequality, $x+3y+3 \leq 0$. This gives

$$0+3\times 0+3=3$$

and

$$3 \nleq 0.$$

That is, it does not satisfy the inequality. Thus all the points below and to the left of the line, as well as those on it (why?), satisfy the inequality.

Exercise 5.4:1
The system of inequalities is

$$\begin{aligned} 2x+y-5 &\leq 0 \\ x-2 &\leq 0 \\ y-4 &\leq 0 \\ x &\geq 0 \\ y &\geq 0. \end{aligned}$$

The solution of the first inequality is the region on, below and to the left of the line

$$2x+y-5=0.$$

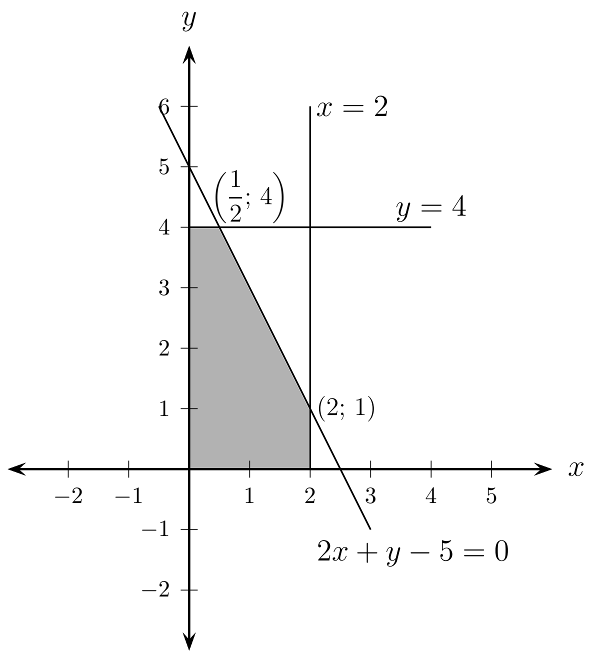
The second implies all x values on or to the left of the line

$$x=2$$

while the third implies all y values on or below the line

$$y=4.$$

The last two inequalities imply the first quadrant, including the axes. The solution is graphed below and indicated by the grey area.



Exercise 5.4:2
The inequality is

$$3x - 7 \leq 5x + 2.$$

Add +7:

$$3x \leq 5x + 9.$$

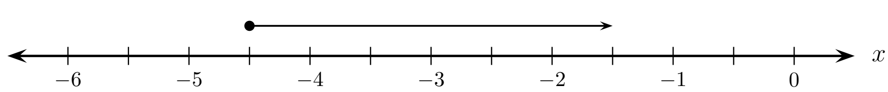
Subtract $5x$:

$$-2x \leq 9.$$

Divide by -2:

$$x \geq -\frac{9}{2}.$$

Thus the solution is all values of x greater or equal to $-\dfrac{9}{2}$ (that is $-4{,}5$), as shown.



Exercise 5.4:3
The inequality is

$$5x + y + 1 < -x - y - 1.$$

Add $x + y + 1$:

$$6x + 2y + 2 < 0.$$

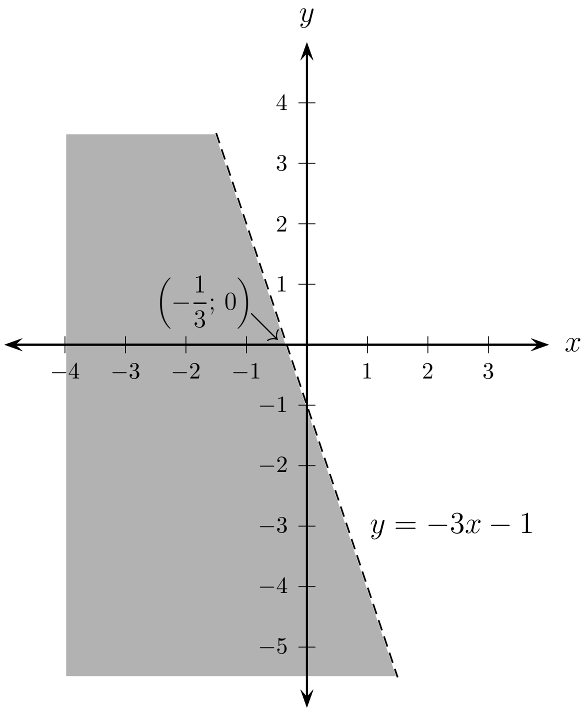
The corresponding straight line is

$$6x + 2y + 2 = 0$$

or

$$y = -3x - 1.$$

Graphically it is represented in the following graph.



The point $(0; 0)$ does not satisfy the inequality. Substitute $(0; 0)$ into the left-hand side of the inequality, $6x + 2y + 2 < 0$. That is,

$$6 \times 0 + 2 \times 0 + 2 = 2$$

and

$$2 \not< 0.$$

Thus, all the points to the left and below the line satisfy the inequality, but the points on the line are excluded. (Why?)

Component 6. An application of differentiation

Exercise 6.1:1
The function is

$$f(x) = 6x.$$

The derivative of $6x$ is the derivative of $6 \times x^1$, which is

$$\begin{aligned} 6 \times 1x^{1-1} &= 6 \times x^0 \\ &= 6 \times 1 \\ &= 6. \end{aligned}$$

Remember that any number to the power of zero equals one. Thus

$$f'(x) = 6.$$

Exercise 6.1:2
The function is

$$f(x) = 3 + 5x.$$

The derivative of 3, which is a constant, is zero.
The derivative of $5x$ is the derivative of $5 \times x^1$, which is equal to

$$\begin{aligned} 5 \times x^{1-1} &= 5 \times x^0 \\ &= 5 \times 1 \\ &= 5. \end{aligned}$$

Thus

$$\begin{aligned} f'(x) &= 0 + 5 \\ &= 5. \end{aligned}$$

Notice that $f(x) = 3 + 5x$ is nothing more than the familiar straight line or linear function. Its intercept on the y axis is 3 and its slope is 5. Its derivative is also 5, which therefore confirms that the derivative represents the slope.

Exercise 6.1:3
The function is

$$f(x) = 3 + 2x^2.$$

The derivative of 3, which is a constant, is zero.
The derivative of x^2 is

$$\begin{aligned} 2x^{2-1} &= 2x^1 \\ &= 2x. \end{aligned}$$

The derivative of $2x^2$ is

$$\begin{aligned} 2 \times 2x^{2-1} &= 2 \times 2x^1 \\ &= 4x. \end{aligned}$$

Thus

$$f'(x) = 4x.$$

Exercise 6.1:4
The profit function is

$$P(x) = 5x - \frac{x^2}{200} - 450.$$

The derivatives of the terms of the profit function are

$$\begin{aligned}
\frac{d}{dx}5x &= 5 \times x^{1-1} \\
&= 5x^0 \\
&= 5 \\
\frac{d}{dx}\left(-\frac{x^2}{200}\right) &= -\frac{2x}{200} \\
&= -\frac{x}{100} \\
\frac{d}{dx}(-450) &= 0
\end{aligned}$$

The derivative of the profit function is

$$P'(x) = 5 - \frac{x}{100}.$$

The marginal profit when 450 knives are produced is

$$\begin{aligned}
P'(450) &= 5 - \frac{450}{100} \\
&= 0{,}5.
\end{aligned}$$

The sale of one additional knife will yield an additional profit of R0,50.

The marginal profit when 750 knives are produced, is

$$\begin{aligned}
P'(750) &= 5 - \frac{750}{100} \\
&= -2{,}5.
\end{aligned}$$

The marginal profit is negative which indicates a decrease in profit. Selling one additional knife will yield a decrease in profit of R2,50.

Exercise 6.1:5
The profit function is

$$P(x) = R(x) - C(x).$$

Thus

$$\begin{aligned}
P(x) &= 10x - \frac{x^2}{1\,000} - (7\,000 + 2x) \\
&= 10x - \frac{x^2}{1\,000} - 7\,000 - 2x \\
&= 8x - \frac{x^2}{1\,000} - 7\,000.
\end{aligned}$$

The derivative of $P(x)$ is

$$P'(x) = 8 - \frac{x}{500}.$$

The marginal profit when 2 000 scissors are produced is

$$\begin{aligned} P'(2\,000) &= 8 - \frac{2\,000}{500} \\ &= 4. \end{aligned}$$

The marginal profit when 4 000 scissors are produced is

$$\begin{aligned} P'(4\,000) &= 8 - \frac{4\,000}{500} \\ &= 0. \end{aligned}$$

The marginal profit when 5 000 scissors are produced is

$$\begin{aligned} P'(5\,000) &= 8 - \frac{5\,000}{500} \\ &= -2. \end{aligned}$$

When 2 000 scissors are produced, an increase in production will yield an increase of R4,00 per pair of scissors in profit.

When 4 000 scissors are produced, the marginal profit is 0 which indicates that 4 000 is the number of scissors that should be produced to maximise profit.

When 5 000 scissors are produced the marginal profit is negative. This indicates that the profit will decrease by R2,00 per pair of scissors.

Exercise 6.2:1
The cost function is

$$C(x) = 575 + 25x - \frac{x^2}{4}.$$

Then

$$\begin{aligned} \frac{d}{dx}(575) &= 0 \\ \frac{d}{dx}(25x) &= 25 \times 1 \times x^{1-1} \\ &= 25 \times x^0 \\ &= 25 \times 1 \\ &= 25 \\ \frac{d}{dx}\left(-\frac{x^2}{4}\right) &= -\frac{2 \times x^{2-1}}{4} \\ &= -\frac{x}{2} \end{aligned}$$

The marginal cost to manufacture x boats is

$$C'(x) = 25 - \frac{x}{2}.$$

Exercise 6.2:2
If 40 boats are manufactured, the marginal cost is

$$\begin{aligned} C'(40) &= 25 - \frac{40}{2} \\ &= 5. \end{aligned}$$

At a production level of 40 boats, the cost to manufacture one additional boat is R5 000,00.

Exercise 6.2:3
If 30 boats are manufactured, the marginal cost is

$$\begin{aligned} C'(30) &= 25 - \frac{30}{2} \\ &= 10. \end{aligned}$$

At a production level of 30 boats it costs R10 000,00 to manufacture one additional boat.

Component 7. Mathematics of finance

Exercise 7.1:1
The interest rate is expressed "per annum" and we have to express the term, T, in years. The following information is given:

$P = 5\,000$

$T = 90 \text{ days} = \frac{90}{365} = \frac{18}{73}$ year

$R = 12\% = 0{,}12$

The interest is calculated as

$$\begin{aligned} I &= PRT \\ &= 5\,000 \times 0{,}12 \times \frac{18}{73} \\ &= 147{,}95 \end{aligned}$$

and the sum accumulated as

$$\begin{aligned} S &= P + I \\ &= 5\,000 + 147{,}95 \\ &= 5\,147{,}95. \end{aligned}$$

The simple interest is R147,95 and the accumulated sum is R5 147,95.

Note that, as I pointed out above, that reference is occasionally made to a 360-day year. This has its origin in pre-calculator days when sums of the above type were tedious. In the above example, the effect of this would be that $T = \frac{90}{360} = \frac{1}{4}$, which obviously makes manual calculation a lot easier. However, unless informed otherwise, you should assume that the "exact" year (365 days) is

used.
(*See Tutorial Letter 101: Using recommended calculator.*)

Exercise 7.1:2
It is given that:

$I = 300$

$T = 1\frac{1}{2}$ year
$R = 12\frac{1}{2}\% = 0{,}125$
From

$$I = PRT$$

we have that

$$\begin{aligned} P &= \frac{I}{RT} \\ &= \frac{300}{0{,}125 \times 1{,}5} \\ &= \frac{300}{0{,}1875} \\ &= 1\,600. \end{aligned}$$

The principal required is R1 600.

Exercise 7.1:3
It is given that:

$P = 3\,000$
$T = 6$ months $= \frac{6}{12} = 0{,}5$ year
$R = 12\% = 0{,}12$
The amount is

$$\begin{aligned} S &= P + I \\ &= P + PRT \\ &= P \times (1 + RT) \\ &= 3\,000 \times (1 + 0{,}12 \times 0{,}5) \\ &= 3\,000 \times 1{,}06 \\ &= 3\,180. \end{aligned}$$

The total amount to be paid back is R3 180.

Exercise 7.1:4
It is given that:

$P = 1\,000$
$T = 4$ years
$R = 10\% = 0{,}10$
The interest is

$$\begin{aligned} I &= PRT \\ &= 1\,000 \times 0{,}10 \times 4 \\ &= 400. \end{aligned}$$

The total interest received is R400.

The amount received per month is calculated as

$$\begin{aligned} \frac{400}{12 \times 4} &= \frac{400}{48} \\ &= 8{,}33. \end{aligned}$$

If the interest is paid monthly, you will receive R8,33 per month.

Exercise 7.1:5
It is given that:

$P = 800$
$S = 823$
$T = 3$ months $= \frac{3}{12} = 0{,}25$ year
From

$$S = P(1 + RT)$$

we have that

$$\begin{aligned}
\frac{S}{P} &= 1 + RT \\
\frac{S}{P} - 1 &= RT \\
\frac{S}{PT} - \frac{1}{T} &= R \\
R &= \frac{823}{800 \times 0{,}25} - \frac{1}{0{,}25} \\
&= \frac{823}{200} - \frac{1}{0{,}25} \\
&= 4{,}115 - 4 \\
&= 0{,}115.
\end{aligned}$$

The interest rate is 11,5% per annum.

Alternatively, the interest earned is

$$\begin{aligned}
I &= 823 - 800 \\
&= 23.
\end{aligned}$$

Then

$$\begin{aligned}
I &= PRT \\
R &= \frac{I}{PT} \\
R &= \frac{23}{800 \times 0{,}25} \\
&= 0{,}115.
\end{aligned}$$

The percentage interest is then $0{,}115 \times 100 = 11{,}5\%$.

Exercise 7.2:1
It is given that:

$S = 4\,000$

$d = 0{,}10$

$T = 6$ months $= \frac{6}{12} = \frac{1}{2}$ year

The discount is calculated as

$$\begin{aligned} D &= SdT \\ &= 4\,000 \times 0{,}10 \times \frac{1}{2} \\ &= 200. \end{aligned}$$

That is, the simple discount is R200.

The discounted value is calculated as

$$\begin{aligned} P &= S - D \\ &= 4\,000 - 200 \\ &= 3\,800. \end{aligned}$$

One will therefore receive R3 800.

Since the interest, I, paid is R200, we use

$$I = PRT$$

to get

$$\begin{aligned} 200 &= 3\,800 \times R \times \frac{1}{2} \\ R &= 2 \times \frac{200}{3\,800} \\ &= 0{,}1053. \end{aligned}$$

Thus, the equivalent simple interest rate is 10,53% per annum.

Exercise 7.2:2(a)
It is given that:

$S = 100$

$d = 0{,}12$

$T = 3 \text{ months} = \frac{3}{12} = \frac{1}{4}$ year

The discount is calculated as

$$\begin{aligned} D &= SdT \\ &= 100 \times 0{,}12 \times \frac{1}{4} \\ &= 3. \end{aligned}$$

That is, the simple discount is R3.

The discounted value is calculated as

$$\begin{aligned} P &= S - D \\ &= 100 - 3 \\ &= 97. \end{aligned}$$

One will therefore receive R97.

Since the interest, I, paid is R3, we use

$$I = PRT$$

to get

$$\begin{aligned} 3 &= 97 \times R \times \frac{1}{4} \\ R &= 4 \times \frac{3}{97} \\ &= 0{,}1237. \end{aligned}$$

Thus, the equivalent simple interest rate is 12,37% per annum.

Exercise 7.2:2(b)
It is given that:

$S = 100$

$d = 0{,}12$

$T = 9 \text{ months} = \frac{9}{12} = \frac{3}{4}$ year

The discount is calculated as

$$\begin{aligned} D &= SdT \\ &= 100 \times 0{,}12 \times \frac{3}{4} \\ &= 9. \end{aligned}$$

That is, the simple discount is R9.

The discounted value is calculated as

$$\begin{aligned} P &= S - D \\ &= 100 - 9 \\ &= 91. \end{aligned}$$

One will therefore receive R91.

Since the interest, I, paid is R9, we use

$$I = PRT$$

to get

$$\begin{aligned} 9 &= 91 \times R \times \frac{3}{4} \\ R &= \frac{4}{3} \times \frac{9}{91} \\ &= 0{,}1319. \end{aligned}$$

Thus, the equivalent simple interest rate is 13,19% per annum.

Exercise 7.2:3
It is given that:

$P = 750$

$d = 0{,}16$

$T = 10 \text{ months} = \frac{10}{12} = \frac{5}{6}$ year

From

$$P = S \times (1 - dT)$$

we calculate the future value of the loan as

$$\begin{aligned} S &= P \div (1 - dT) \\ &= \frac{750}{\left(1 - 0{,}16 \times \dfrac{5}{6}\right)} \\ &= 865{,}38. \end{aligned}$$

Mary will have to repay June R865,38 in ten months' time.

Exercise 7.3:1
It is given that:

$P = 1\,000$

$R = \dfrac{0{,}08}{2} = 0{,}04$ because it is compounded biannually

$T = 2 \times 1 = 2$ half years

Then

$$\begin{aligned} S &= P\,(1 + R)^T \\ &= 1\,000\left(1 + \frac{0{,}08}{2}\right)^2 \\ &= 1\,000 \times 1{,}04^2 \\ &= 1\,081{,}60. \end{aligned}$$

The compounded amount is R1 081,60.

The interest earned is

$$1\,081{,}60 - 1\,000 = 81{,}60.$$

The interest earned is R81,60.

Exercise 7.3:2
It is given that:

$P = 2\,000$

$R = \dfrac{0{,}12}{4} = 0{,}03$ because it is compounded quarterly

$T = 2\dfrac{1}{2} \times 4 = 10$ quarters

Then

$$\begin{aligned} S &= P(1 + R)^T \\ &= 2\,000(1 + 0{,}03)^{10} \\ &= 2\,000 \times 1{,}03^{10} \\ &= 2\,687{,}83. \end{aligned}$$

The total amount available is R2 687,83.

Exercise 7.3:3(a)
In all three cases we have $P = 1\,000$.

We have that:

$T = 2$

$R = 0{,}10$ simple interest

Then

$$\begin{aligned} I &= P \times R \times T \\ &= 1\,000 \times 0{,}10 \times 2 \\ &= 200. \end{aligned}$$

The interest earned is R200.

Exercise 7.3:3(b)
We have that:

$T = 2 \times 2 = 4$ half years

$R = \dfrac{0{,}095}{2}$

Then

$$\begin{aligned} S &= P(1+R)^T \\ &= 1\,000\left(1 + \frac{0{,}095}{2}\right)^4 \\ &= 1\,203{,}97. \end{aligned}$$

The interest earned is R203,97.

Exercise 7.3:3(c)
We have that:

$T = 2 \times 4 = 8$ quarters

$R = \dfrac{0{,}09}{4}$

Then

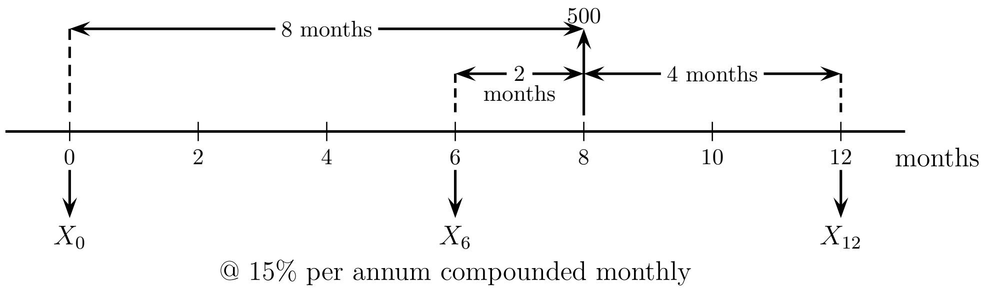
$$\begin{aligned} S &= 1\,000\left(1 + \frac{0{,}09}{4}\right)^8 \\ &= 1\,194{,}83. \end{aligned}$$

The interest earned is R194,83.

The best investment is $9\frac{1}{2}\%$ interest per year, compounded biannually.

Exercise 7.4:1
The three problems can be represented as follows on a time line (all at 15% interest per annum,

compounded monthly).



Exercise 7.4:1(a)
We have to determine the size of payment X_0. From month zero to month eight there are eight months. We have to discount the debt eight months backwards to determine the present value. From

$$S = P(1+R)^T$$

we have that

$$\begin{aligned} P &= S \div (1+T)^T \\ &= 500 \div \left(1 + \frac{0{,}15}{12}\right)^8 \\ &= 452{,}70. \end{aligned}$$

The single payment that will repay her debt now (X_0) is R452,70.

Exercise 7.4:1(b)
We have to determine the size of payment X_6. From month six to month eight there are two months. We have to discount the debt two months backwards to determine the present value:

$$\begin{aligned} P &= 500 \div \left(1 + \frac{0{,}15}{12}\right)^2 \\ &= 487{,}73. \end{aligned}$$

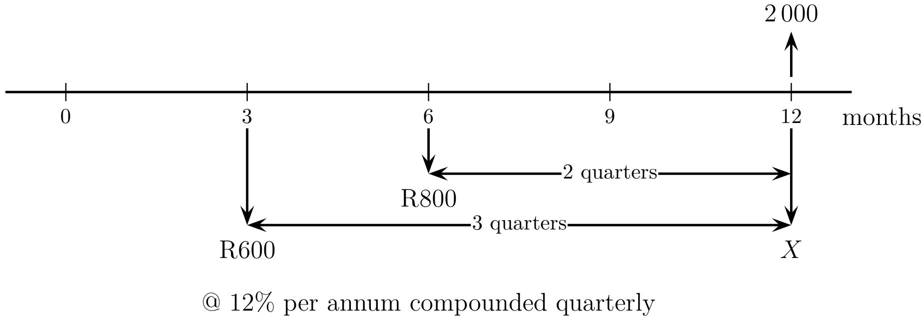
The single payment that will repay her debt six months from now (X_6) is R487,73.

Exercise 7.4:1(c)
We have to determine the size of payment X_{12}. From month eight to one year there are four months. We must move the debt forward four months:

$$\begin{aligned} S &= 500 \times \left(1 + \frac{0{,}15}{12}\right)^4 \\ &= 525{,}47. \end{aligned}$$

The single payment that will repay her debt one year from now (X_{12}) is R525,47.

Exercise 7.4:2
In the following diagram, debts are shown above the line and payments below.



Payments: The numerical values of the R600 and R800 payments must be moved to the end of the year. There is also a Rx payment at month twelve.

For the R600 payment:

The money is moved forward nine months (ie three quarters). The future value is

$$\begin{aligned} S &= 600 \times \left(1 + \frac{0{,}12}{4}\right)^3 \\ &= 655{,}64. \end{aligned}$$

The value of the R600 payment at the end of the year is R655,64.

For the R800 payment:

The money is moved forward six months (ie two quarters). The future value is

$$\begin{aligned} S &= 800 \times \left(1 + \frac{0{,}12}{4}\right)^2 \\ &= 848{,}72. \end{aligned}$$

The value of the R800 payment at the end of the year is R848,72.

For the Rx payment:

As this is the last payment no interest is involved and the payment remains Rx.

Debt: There is only one debt, namely R2 000 at the end of the year.

For the R2 000 debt:

No interest is involved for the debt of R2 000 at the end of the year.

Remember, the total amount to be paid is equal to the total debt. Therefore

$$\begin{aligned} \text{payments} &= \text{debt} \\ 655{,}64 + 848{,}72 + x &= 2\,000 \\ 1\,504{,}36 + x &= 2\,000 \\ x &= 2\,000 - 1\,504{,}36 \\ &= 495{,}64. \end{aligned}$$

She must thus pay R495,64 at the end of the year to settle her debt.

Exercise 7.5
(*See Tutorial Letter 101: Using the recommended calculator.*)

Exercise 7.5:1
The accumulated amount is

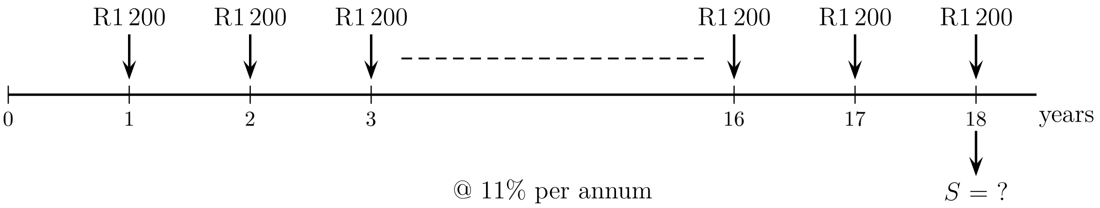
$$\begin{aligned} S &= S_1 + S_2 + S_3 + S_4 + S_5 \\ &= 600\,(1{,}1)^4 + 600\,(1{,}1)^3 + 600\,(1{,}1)^2 + 600\,(1{,}1)^1 + 600 \\ &= 3\,663{,}06. \end{aligned}$$

The accumulated amount is thus R3 663,06.

Alternatively, you could have used the formula for the future value of the annuity, giving

$$\begin{aligned} S &= R\, s_{\overline{n}|i} \\ &= R\left[\frac{(1+i)^n - 1}{i}\right] \\ &= 600\left[\frac{(1+0{,}10)^5 - 1}{0{,}10}\right] \\ &= \frac{600(1{,}1^5 - 1)}{0{,}10} \\ &= 3\,663{,}06. \end{aligned}$$

Exercise 7.5:2
The time line is given below.



It is given that:

$R = 1\,200$

$i = 0{,}11$ per year

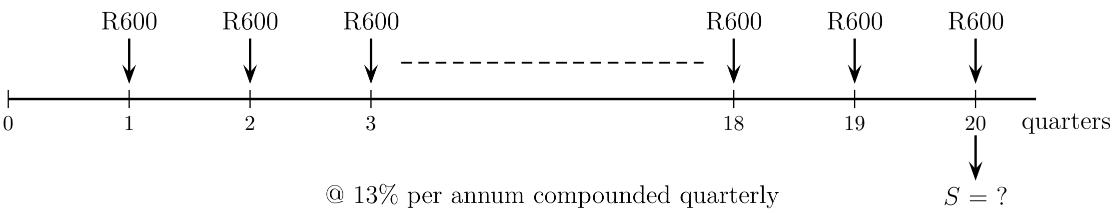
$n = 18$ years

Thus

$$\begin{aligned} S &= 1\,200\, s_{\overline{18}|0{,}11} \\ &= 1\,200\left[\frac{(1+i)^n - 1}{i}\right] \\ &= 1\,200\left[\frac{1{,}11^{18} - 1}{0{,}11}\right] \\ &= 60\,475{,}12. \end{aligned}$$

When her daughter is 18 years old, the amount will be R60 475,12.

Exercise 7.5:3
The time line is given below.



It is given that:

$R = 600$

$i = \dfrac{0{,}13}{4} = 0{,}0325$ per quarter

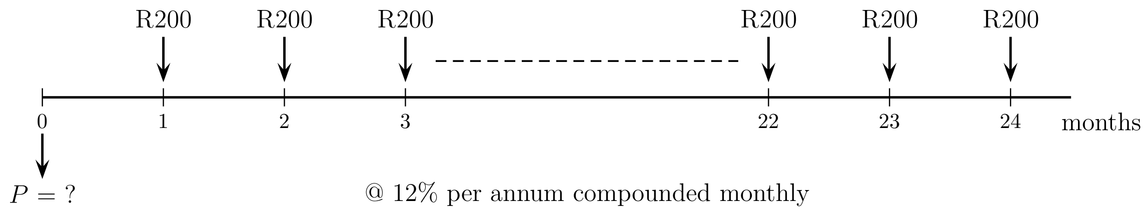
$n = 5 \times 4 = 20$ quarters

Thus

$$\begin{aligned} S &= 600\, s_{\overline{20}|0{,}0325} \\ &= 600\left[\frac{1{,}0325^{20} - 1}{0{,}0325}\right] \\ &= 16\,538{,}55. \end{aligned}$$

Thus, the accumulated amount at the end of the term is R16 538,55.

Exercise 7.5:4
The time line is given below.



It is given that:

$R = 200$

$i = \dfrac{0{,}12}{12} = 0{,}01$ per month

$n = 24$ months

The present value of the 24 payments is

$$\begin{aligned} P &= 200\, a_{\overline{24}|0{,}01} \\ &= 200\left[\frac{(1+i)^n - 1}{i\,(1+i)^n}\right] \\ &= 200\left[\frac{1{,}01^{24} - 1}{0{,}01 \times 1{,}01^{24}}\right] \\ &= 4\,248{,}68. \end{aligned}$$

Thus the present value of the payments is R4 248,68. The cost of the motorbike is calculated as

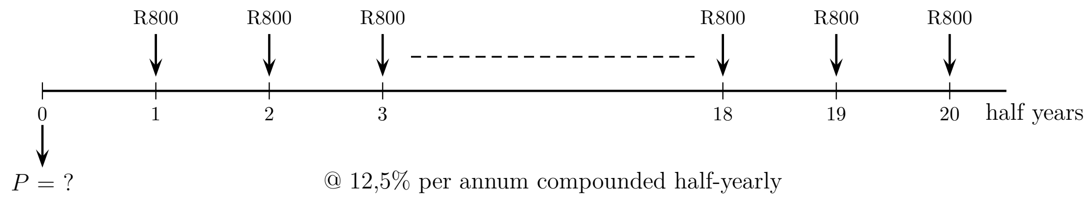
$$1\,000 + 4\,248{,}68 = 5\,248{,}68.$$

The cost of the motorbike is R5 248,68. The interest paid is the difference between the sum of all payments and the cost, namely

$$\begin{aligned} I &= (1\,000 + 24 \times 200) - 5\,248{,}68 \\ &= 551{,}32. \end{aligned}$$

The interest that he paid is R551,32.

Exercise 7.5:5
The relevant time line is given below.



It is given that:

$R = 800$

$i = \dfrac{0{,}125}{2} = 0{,}0625$ per half-year

$n = 10 \times 2 = 20$ half-years

The present value is thus

$$\begin{aligned} P &= 800\, a_{\overline{20}|0{,}0625} \\ &= 800 \left[\frac{1{,}0625^{20} - 1}{0{,}0625 \times 1{,}0625^{20}} \right] \\ &= 8\,992{,}58. \end{aligned}$$

Thus, the amount of R8 992,58 must be invested now at 12,5% per annum, compounded half-yearly, to provide half-yearly payments of R800 for ten years.

Exercise 7.6:1
The present value of the loan is R225 000 (ie 270 000 − 45 000).

It is given that:

$P = 225\,000$

$i = \dfrac{0{,}115}{12}$ per month

$n = 12 \times 20 = 240$ months

The payments are

$$\begin{aligned} P &= R\, a_{\overline{n}|i} \\ 225\,000 &= R\, a_{\overline{240}|\frac{0{,}115}{12}} \\ R &= 225\,000 \div a_{\overline{240}|\frac{0{,}115}{12}} \\ &= 2\,399{,}47. \end{aligned}$$

The monthly payments are R2 399,47.

Exercise 7.6:2
It is given that:

$P = 60\,000$

$i = 0{,}12$ per year

$n = 10$ years

The payments are

$$\begin{aligned} P &= R\, a_{\overline{n}|i} \\ 60\,000 &= R\, a_{\overline{10}|0{,}12} \\ R &= 60\,000 \div a_{\overline{10}|0{,}12} \\ &= 10\,619{,}05. \end{aligned}$$

Thus, you will receive R10 619,05 at the end of each year for ten years.

Exercise 7.6:3
It is given that:

$P = 4\,000$

$i = \dfrac{0{,}15}{2} = 0{,}075$ per half year

$n = 3 \times 2 = 6$ half years

The payments are

$$\begin{aligned} R &= 4\,000 \div a_{\overline{6}|0{,}075} \\ &= 852{,}18. \end{aligned}$$

The payments are R852,18.

The amortisation schedule is as follows:

Period (that is half-years)	Outstanding principal at half-year beginning	Interest due at end of half-year	Payment	Principal repaid
1	4 000,00	300,00	852,18	552,18
2	3 447,82	258,59	852,18	593,59
3	2 854,23	214,07	852,18	638,11
4	2 216,12	166,21	852,18	685,97
5	1 530,15	114,76	852,18	737,42
6	792,73	59,45	852,18	792,73
Total		1 113,08	5 113,08	4 000,00